Existence and Uniqueness of Global Mild Solutions for Nonlocal Cauchy Systems in Banach Spaces.

Yangari M.*

*Escuela Politécnica Nacional, Departamento de Matemática, Ladrón de Guevara E11-253, Quito, Ecuador

e-mail: miguel.yangari@epn.edu.ec

Resumen: El objetivo de este artículo es estudiar la existencia y unicidad de soluciones de sistemas no-locales cuando los términos de difusión están dados por un generador infinitesimal de operadores fuertemente continuos de operadores lineales acotados.

Palabras claves: Sistemas no-lineales, generador infinitesimal, operadores fuertemente continuos, soluciones débiles.

Abstract: The aim of this paper is to study the existence and uniqueness of solutions to nonlocal systems when the diffusion terms are given by infinitesimal generators of strongly continuous semigroups of bounded linear operators.

Keywords: Nonlinear system, infinitesimal operator, strongly continuous semigroup, mild solution.

1. INTRODUCTION

In this paper we discuss the existence and uniqueness of solutions $u = (u_i)_{i=1}^m$ with $m \in \mathbb{N}^*$, to the nonlocal system:

$$\begin{cases} \partial_t u + Au &= F(t,u), \quad \forall\, t \in (0,\infty) \\ u(0) &= u_0, \end{cases} \tag{1}$$

where $A = \text{diag}(A_1, ..., A_m)$ with A_i the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T_{t,i}$ in the Banach space X for all $i \in [\![1,m]\!] := \{1,...,m\}$, $u_0 \in X^m$ and $F : [0,\infty) \times X^m \to X^m$ a given valued function, where X^m is the Banach product space doted with the norm $\|u\|_{X^m} = \sum_{i=1}^m \|u_i\|_X$.

The above nonlocal problem (1) with $m = 1$ has been studied extensively. Byszewski and Lasmikanthem [3], [4], [6] give the existence and uniqueness of mild solutions when F satisfies locally Lipschitz-type conditions. Cabré and Roquejoffre in [7] state the existence of global classical solutions when the reaction term satisfies globally Lipschitz conditions. In [9] Lin and Liu discuss the semilinear integro-differential equations under Lipschitz-type conditions. Ntougas and Tsamatos [10], [11] study the case of compactness conditions on T_t. Byszewski and Akca [5] give the existence of functional-differential equation when T_t is compact. Benchohra and Ntouyas [2] discuss the second order differential equations with nonlocal conditions under compact conditions. In [8] Fu and Ezzinbi study the neutral functional differential equations with nonlocal initial conditions. Aizicovici and McKibben [1] give the existence of integral solutions of nonlinear differential inclusions with nonlocal conditions.

The work on the single equation can be extended to the system (1), thus, as general assumptions, the reaction term $F = (f_i)_{i=1}^m$ satisfies for all $i \in [\![1,m]\!]$:

$$\begin{aligned} & f_i \in C^1([0,+\infty) \times X^m; X), \\ & f_i(t, \cdot\,) \text{ is globally Lipschitz in } X^m \text{ uniformly in } t \geq 0, \end{aligned} \tag{2}$$

Our main result states the existence and uniqueness of mild solutions, where, by a mild solution of the nonlocal system (1) we mean the function $u \in C([0,\infty), X)^m$ which satisfies

$$u(t) = \mathtt{T}_t u_0 + \int_0^t \mathtt{T}_{t-s} F(s, u(s)) ds \tag{3}$$

where $\mathtt{T}_t = diag(T_{t,1}, ..., T_{t,m})$.

Theorem 1.1 *Let X be a Banach space, F satisfies (2) and the initial condition u_0 belongs to X^m. Then, the system (1) has a unique mild solution $u \in C([0,\infty), X)^m$.*

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As a corollary of this theorem, we see that the unique mild solution can be obtained by an iterative process.

Corollary 1.1 *The mild solution u of (1) can be obtained as the limit in the $C([0,\infty),X)^m$-norm of the sequence $(u^n)_{n\in\mathbb{N}}$, where:*

$$u^{n+1}(t) = \mathtt{T}_t u_0 + \int_0^t \mathtt{T}_{t-s} F(s, u^n(s)) ds$$

with $u^0(t) = \mathtt{T}_t u_0$.

Finally, to state the following result of existence, we assume a perturbation of the nonlinear term. Hence:

Lemma 1.1 *If u is the mild solution of the system (1) with $u_0 \in X^m$, F satisfies (2) for all $i \in [\![1,m]\!]$. Then, for any $l \in \mathbb{R}$, $\tilde{u}(t) = e^{lt}u(t)$ is the mild solution of the system (1) with $u_0 \in X^m$ and $F(t,u)$ replaced by $\widetilde{F}(t,\tilde{u}) = l\tilde{u} + e^{lt}F(t, e^{-lt}\tilde{u})$.*

2. THE SEMIGROUP AND ITS GENERATOR

Taking $i \in [\![1,m]\!]$ fixed throughout this short section, we remember that a family $T_{t,i}$ with $0 \le t < \infty$ of bounded linear operators from X into X is a semigroup on the Banach space X if

i) $T_{0,i} = I$, where I is the identity operator on X.

ii) $T_{t+s,i} = T_{t,i}T_{s,i}$ for every $t, s \ge 0$

A semigroup of bounded linear operators, $T_{t,i}$, is uniformly continuous if

$$\lim_{t\downarrow 0} \|T_{t,i} - I\| = 0$$

The linear operator A_i defined by

$$D(A_i) = \left\{ x \in X : \lim_{t\downarrow 0} \frac{T_{t,i}x - x}{t} \ \text{ exists} \right\}$$

and

$$-A_i x = \lim_{t\downarrow 0} \frac{T_{t,i}x - x}{t} = \frac{dT_{t,i}x}{dt} |_{t=0} \qquad \text{for } x \in D(A_i)$$

is the infinitesimal generator of the semigroup $T_{t,i}$, $D(A_i)$ is the domain of $-A_i$. From the above discussion it is clear, if $T_{t,i}$ is a uniformly continuous semigroup of bounded linear operators, then

$$\lim_{s\downarrow t} \|T_{s,i} - T_{t,i}\| = 0$$

Moreover, a semigroup $T_{t,i}$ with $0 \le t < \infty$, of bounded linear operators on X is a strongly continuous semigroup if

$$\lim_{t\downarrow 0} T_{t,i}x = x, \qquad \text{for every } x \in X$$

Next, a useful fact for future purposes. If $T_{t,i}$ is a strongly continuous semigroup, then there exist constants $M_i \ge 1$ and $\omega_i \ge 0$ such that

$$\|T_{t,i}\| \le M_i e^{\omega_i t} \qquad \text{for } 0 \le t < \infty$$

3. MAIN RESULTS

Proof of Theorem 1.1. Given any $T > 0$, we are interested in the nonlinear problem

$$\left\{ \begin{array}{rcl} \partial_t u + Au & = & F(t,u), \quad in\ (0,T) \\ u(0) & = & u_0, \end{array} \right. \tag{1}$$

where $A = \text{diag}(A_1, ..., A_m)$, $u = (u_i)_{i=1}^m$ and $u_0 \in X^m$. We define the map

$$N_{u_0}(u)(t) := \mathtt{T}_t u_0 + \int_0^t \mathtt{T}_{t-s} F(s, u(s)) ds \tag{2}$$

here $\mathtt{T}_t = \text{diag}(T_{t,1}, ..., T_{t,m})$.

Let us prove, if $u \in C([0,T];X)^m$ then $N_{u_0}(u) \in C([0,T];X)^m$, i.e., $N_{u_0} : C([0,T];X)^m \to C([0,T];X)^m$, indeed, since $u \in C([0,T];X)^m$, then $u \in C([0,T];X^m)$ and since $F_i \in C([0,+\infty) \times X^m; X)$, we have that $F_i(\,\cdot\,, u) \in C([0,+\infty);X)$. Now, since $u_0 \in X$ and $T > 0$, it is easy to see that $N_{u_0,i}(u) \in C([0,T];X)$.

Furthermore, we claim that N_{u_0} is Lipschitz in $C([0,T];X)^m$. Indeed, let $u, v \in C([0,T];X)^m$, so

$$\begin{aligned} &\left\| N_{u_0,i}(u)(t) - N_{u_0,i}(v)(t) \right\|_X \\ &\quad \le \int_0^t \|T_{t-s,i}(F_i(u(s)) - F_i(v(s)))\|_X \, ds \\ &\quad \le \int_0^t \|T_{t-s,i}\| \, \|F_i(u(s)) - F_i(v(s))\|_X \, ds \\ &\quad \le \overline{M} Lip_u(F_i) \int_0^t \|u(s) - v(s)\|_{X^m} \, ds \\ &\quad = \overline{M} Lip_u(F_i) \int_0^t \sum_{j=1}^m \left\| u_j(s) - v_j(s) \right\|_X ds \\ &\quad \le t\overline{M} Lip_u(F_i) \, \|u - v\|_{C([0,T];X)^m} \end{aligned}$$

where $\overline{M} = \sup_{t\in[0,T], i\in[\![1,m]\!]} \|T_{t,i}\|$, we recall that for any strongly continuous semigroup, we have that $\|T_{t,i}\| \le Ce^{\omega_i t}$ for some constants C and ω_i. So, from the above computations, taking the supremum in $[0,T]$ and adding in $i \in [\![1,m]\!]$, we have that

$$\begin{aligned} &\|N_{u_0}(u) - N_{u_0}(v)\|_{C([0,T];X)^m} \\ &\qquad \le T\overline{M} \left[\sum_{i=1}^m Lip_u(F_i) \right] \|u - v\|_{C([0,T];X)^m} \end{aligned}$$

thus N_{u_0} is Lipschitz with constant $\overline{M}T\,[\sum_{i=1}^m Lip_u(F_i)]$. Now, it follows by induction that $(N_{u_0})^k$ is Lipschitz in $C([0,T];X)^m$ with Lipschitz constant $\{\overline{M}T\,[\sum_{i=1}^m Lip_u(F_i)]\}^k/k!$, where k is any positive integer. This constant is less than 1 if we take k large enough. Then, we conclude that N_{u_0} has a unique fixed point and therefore, there exists a unique solution u that satisfies $u = N_{u_0}(u)$ for every $T > 0$. Given $0 < T < T'$, the mild solution in $(0, T')$ must coincide in $(0, T)$ with the mild solution in this interval, by uniqueness. Thus, under assumption (2), the mild solution of (1) extends uniquely to all $t \in [0, +\infty)$, i.e., it is global in time.

Proof of Corolario 1.1. Let define the sequence $(u^i)_{i\in\mathbb{N}}$ given by

$$u^i = M^i(u_0) = M(M^{i-1}(u_0))$$

where $M = N_{u_0}^m$ with $m \in \mathbb{N}$ such that the Lipschitz constant founded in Theorem 1.1 satisfies

$$\alpha := \left\{\overline{M}T\left[\sum_{i=1}^m Lip_u(G_i)\right]\right\}^m /m! < 1$$

In what follows all the norms will be in $C([0,T];X)^m$. Now, we claim that (M^i) is a Cauchy sequence, indeed, given $\varepsilon > 0$, we need to find $\Lambda > 0$ such that

$$\left\|M^n(u_0) - M^k(u_0)\right\| < \varepsilon, \qquad \forall\, n,k \geq \Lambda$$

Hence,

$$\begin{aligned}
&\left\|M^n(u_0) - M^k(u_0)\right\| \\
&\leq \alpha^k \left\|M^{n-k}(u_0) - u_0\right\| \\
&\leq \alpha^k(\left\|M^{n-k}(u_0) - M^{n-k-1}(u_0)\right\| + ... \\
&\qquad ... + \|M(u_0) - u_0\|) \\
&\leq \alpha^k(\alpha^{n-k-1}\|M(u_0) - u_0\| + ... + \|M(u_0) - u_0\|) \\
&= \alpha^k \frac{1-\alpha^{n-k}}{1-\alpha}\|M(u_0) - u_0\| \\
&\leq \frac{\alpha^k}{1-\alpha}\|M(u_0) - u_0\| < \varepsilon
\end{aligned}$$

if $n \geq k \geq \Lambda$ with $\Lambda > 0$ large enough. Thus, we have that $(M^i)_{i\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $C([0,T];X)^m$, therefore, there exists $u \in C([0,T];X)^m$ such that

$$u = \lim_{i\to\infty} M^i(u_0), \qquad as\ i \to +\infty \tag{3}$$

Now, we prove that u founded above also satisfies

$$u = \lim_{j\to\infty} N_{u_0}^j(u_0), \qquad as\ j \to +\infty \tag{4}$$

To do this, we note that the limit u in (3) does not depend of the initial condition, i.e., for any $w \in C([0,T];X)^m$ we have $M^i(w) \to u$, indeed,

$$\begin{aligned}
\left\|M^i(w) - u\right\| &\leq \left\|M^i(w) - M^i(u_0)\right\| + \left\|M^i(u_0) - u\right\| \\
&\leq \alpha^i\|w - u_0\| + \left\|M^i(u_0) - u\right\| \to 0
\end{aligned}$$

since $\alpha < 1$ and by (3). Now, in order to prove (4), we note that for all $j \in \mathbb{N}$, it is possible to find $k \in [0, m[$ such that $j = k + mi$ for some $i \in \mathbb{N}$, hence

$$\begin{aligned}
\left\|N_{u_0}^j(u_0) - u\right\| &= \left\|N_{u_0}^{k+mi}(u_0) - u\right\| \\
&= \left\|N_{u_0}^{mi}(N_{u_0}^k(u_0)) - u\right\| \\
&= \left\|M^i(N_{u_0}^k(u_0)) - u\right\| \\
&= \left\|M^i(w) - u\right\| \to 0
\end{aligned}$$

with $w = N_{u_0}^k(u_0) \in C([0,T];X)^m$, as $j \to +\infty$. To conclude the proof, we claim that u is the unique mild solution stated in Theorem 1.1, for which, by uniqueness, we only need to verify that $u = N_{u_0}(u)$, indeed,

$$\begin{aligned}
&\|N_{u_0}(u) - u\| \\
&\quad= \left\|N_{u_0}(u) - N_{u_0}(N_{u_0}^{i-1}(u_0))\right\| + \left\|N_{u_0}^i(u_0) - u\right\| \\
&\quad\leq \overline{M}T\left[\sum_{i=1}^m Lip_u(G_i)\right]\left\|u - N_{u_0}^{i-1}(u_0)\right\| \\
&\qquad\qquad + \left\|N_{u_0}^i(u_0) - u\right\| \to 0
\end{aligned}$$

by (4), as $i \to +\infty$.

Proof of Lemma 1.1. Since u satisfies (3), we have for all $i \in [\![1, m]\!]$

$$u_i(s) = T_{s,i}u_{0,i} + \int_0^s T_{s-\tau,i}f_i(\tau, u(\tau))d\tau$$

Hence, for all $s \in [0, t]$

$$T_{t-s,i}u_i(s) = T_{t,i}u_{0,i} + \int_0^s T_{t-\tau,i}f_i(\tau, u(\tau))d\tau$$

We now multiply by le^{ls} and using integration by parts on s,

$$\begin{aligned}
&\int_0^t le^{ls}T_{t-s,i}u_i(s)ds \\
&\quad= \int_0^t le^{ls}T_{t,i}u_{0,i}ds \\
&\qquad\qquad + \int_0^t le^{ls}\int_0^s T_{t-\tau,i}f_i(\tau, u(\tau))d\tau ds \\
&\quad= (e^{lt} - 1)T_{t,i}u_{0,i} + e^{lt}\int_0^t T_{t-\tau,i}f_i(\tau, u(\tau))d\tau
\end{aligned}$$

$$
\begin{aligned}
& - \int_0^t e^{ls} T_{t-s,i} f_i(s,u(s))ds \\
= \quad & e^{lt} u_i(t) - T_{t,i} u_{0,i} \\
& - \int_0^t e^{ls} T_{t-s,i} f_i(s,u(s))ds
\end{aligned}
$$

hence taking $\tilde{u}(t) = e^{lt}u(t)$, we have

$$
\tilde{u}(t) = T_{t,i}u_{0,i} + \int_0^t T_{t-s,i}[l\tilde{u}(s) + e^{ls} f_i(s, e^{-ls}\tilde{u}(s))]ds
$$

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