Abstract: The aim of this paper is to study the existence and uniqueness of global solutions in time to systems of equations, whence the diffusion terms are given by sectorial generators.

Keywords: Reaction diffusion systems, sectorial operators, infinitesimal generators.

1. INTRODUCTION

In this paper, we study the global existence and uniqueness of sectorial solutions to the system

\[
\begin{align*}
\partial_t u_i + A_i u_i &= f_i(u), \quad \forall \ t \in (0, \infty) \\
u_i(0, x) &= u_{i0}(x),
\end{align*}
\]

where \(A_i\) are sectorial operators in the Banach space \((X, \| \cdot \|)\), \(u_{i0} \in X\) and \(f_i : [0, \infty) \times X^m \to X\) a given valued function for all \(i \in [1, m]\) := \{1, ..., m\}, where \(X^m\) is the Banach product space dotted with the norm \(\| u \|_m = \sum_{i=1}^m \| u_i \|\).

In the case \(m = 1\), since work (Byszewski and Lakshmikantham, 1990), (Byszewski, 1991), (Byszewski, 1993), there has been increasing interest in studying abstract problems in Banach Spaces (cf., e.g., (Aizicovici and McKibben, 2000) and references therein). For material intimately related to the present paper, we refer to (Henry, 1981), where is studied the existence of sectorial solutions for the single equations. Also in (Jackson, 1993), (Liang et al, 2002), nonlocal autonomous parabolic problems are investigated, with \(f\) being Lipschitz continuous.

The existence of mild and classical solutions for reaction diffusion equations involving a particular class of sectorial operators (fractional Laplacians) are studied in (Cabrera and Roquejoffre, 2013). See also the result in (Aizicovici and McKibben, 2000), in which \(\{A(t)\}_{0 \leq t \leq T}\) is a family of \(m\)-accretive operators in \(X\) generating a compact evolution family, and the existence of integral solutions to the associated nonlocal problem is shown. In the case \(m > 1\), we refer to (Yangari, 2015) in which is studied the existence and uniqueness of mild solutions of a reaction diffusion system with infinitesimal generators.

In order to improve the notation, we consider the system

\[
\begin{align*}
\partial_t u + A u &= f(t, u), \quad \forall \ t \in (0, \infty) \\
u(0) &= u_0,
\end{align*}
\]

where \(u = (u_i)_{i=1}^m\), \(f = (f_i)_{i=1}^m\) and \(A = \text{diag} (A_1, ..., A_m)\). Moreover, we consider the norm \(\| u \|_a = \sum_{i=1}^m \| u_i \|_{a_i}\) on the space \(X^a = \coprod_i^m X_i^a\), where the space \(X_i^a\) is defined in the next section.

Throughout this paper, we assume \(\Omega\) some open set in \(\mathbb{R} \times X^a\) and \(f_i : \Omega \to X\) is locally Hölder continuous in \(t\) and locally Lipschitz continuous in \(u\) on \(\Omega\) for all \(i \in [1, m]\). More
Let us note that every bounded linear operator on a Banach space is in the resolvent set of its near operator \( \theta \) assume for some constants such that for \((t, u(t)) \in \Omega, u(t) \in \mathcal{D}(A), \Sigma(t) \) exists, \( t \mapsto f(t, u(t)) \) is locally Hölder continuous and

\[
\left\| \int_{t_0}^{t_0+\rho} \|f(t, u(t))\|_m \, dt \right\| < \infty
\]

for some \( \rho > 0 \) and the differential equation (1) is verified.

2. SECTORIAL OPERATORS

Taking \( i \in [1, m] \) fixed throughout this section, we call a linear operator \( A_i \) in a Banach space \( X \), a sectorial operator if it is a closed densely defined operator such that, for some \( \phi \in (0, \pi/2) \), \( M \geq 1 \) and a real \( \varepsilon \), the sector

\[
S_{\varepsilon, \phi} = \{ \lambda \mid \phi \leq |\arg(\lambda - \epsilon)| \leq \pi, \lambda \neq \epsilon \}
\]

is in the resolvent set of \( A_i \) and

\[
\left\| (\lambda I - A_i)^{-1} \right\| \leq M/|\lambda - \epsilon| \quad \text{for all } \lambda \in S_{\varepsilon, \phi}.
\]

Let us note that every bounded linear operator on a Banach space is sectorial. Also, if we define

\[
e^{-A_i t} = \frac{1}{2\pi i} \int_\Gamma (\lambda I + A_i)^{-1} e^{\lambda t} \, d\lambda
\]

where \( \Gamma \) is a contour in the resolvent of \(-A_i\) with arg(\(\lambda\)) \( \pm \theta \) as \( |\lambda| \to \infty \) for some \( \theta \in (\pi/2, \pi) \), we have that \(-A_i\) is the infinitesimal generator of the analytic semigroup \((e^{-A_i(t)})_{t \geq 0}\), moreover, if \( \text{Re}\sigma(A_i) > b_i \), then for \( t > 0 \)

\[
\|e^{-A_i t}\| \leq c e^{-b_i t}, \quad \|A_i e^{-A_i t}\| \leq \frac{c}{t} e^{-b_i t}.
\]

It is important to note that, if \( B \) is a bounded linear operator, then \( e^{-Bt} \) as defined above extends to a group of linear operators and verifies

\[
e^{-Bt} e^{-Bs} = e^{-(t+s)}, \quad \text{for } -\infty < t, s < \infty.
\]

In order to define the fractional power of a sectorial operator \( A_i \), we assume \( \text{Re}\sigma(A_i) > 0 \), so, for any \( \alpha \in (0, 1) \)

\[
A_i^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty \tau^{\alpha-1} e^{-\tau A_i} d\tau.
\]

Taking \( A_i^{-\alpha} \) defined as above, we have that this operator is a bounded linear operator on \( X \) which is one-one and satisfies

\[
A_i^{1-\alpha} A_i^{-\beta} = A_i^{1-(\alpha+\beta)}.
\]

Furthermore, \( A_i^\alpha \) represents the inverse operator of \( A_i^{-\alpha} \) with \( \mathcal{D}(A_i^\alpha) = R(A_i^{-\alpha}) \) and \( A_i^0 \) is the identity on \( X \). An important result concerning positive powers of sectorial operators is

\[
\|A_i^\alpha e^{-A_i t}\| \leq c a t^{-a} e^{-b_i t} \quad \text{for } t > 0
\]

with \( \text{Re}\sigma(A_i) > b_i > 0 \) if \( u \in \mathcal{D}(A_i^\alpha) \)

\[
\left\| (e^{-A_i t} - I) u \right\| \leq \frac{1}{\alpha} \epsilon_0 \| A_i^\alpha u \|
\]

also, \( A_i^\alpha A_i^\beta = A_i^{\beta+\alpha} = A_i^{\alpha+\beta} \) on \( \mathcal{D}(A_i^\gamma) \) with \( \gamma = \max(\alpha, \beta, \alpha + \beta) \).

Now, we consider the fractional powers of \( B_i := A_i + a_i I \) with \( a_i \in \mathbb{R} \) chosen so \( \sigma(B_i) > 0 \), where \( \sigma(B_i) \) is the spectrum of \( B_i \). We define the Banach space \( X_i^\alpha = D(B_i^\alpha) \) with the norm \( \|u\|_{\alpha,i} = \| B_i^\alpha u \| \), where \( D(B_i^\alpha) \) is the domain of the operator \( B_i^\alpha \). Finally, taking \( \alpha \geq \beta \geq 0 \), then \( X_i^\alpha \) is a dense subspace of \( X_i^\beta \) with continuous inclusion, also, \( X_i^0 = X \).

For more information about sectorial operators we refer the reader to (Henry, 1981).

3. MAIN RESULTS

In order to state our first result, since \(-A_i\) is the infinitesimal generator of the analytic semigroup \((e^{-A_i(t)})_{t \geq 0}\) for each \( i \in [1, m] \), we define the weak formulation for the system (1) given by

\[
u(t) = P(t - t_0) u_0 + \int_{t_0}^t P(t - s) f(s, u(s)) \, ds
\]

with

\[
P(t) = \text{diag}(e^{-A_1(t)}, ..., e^{-A_m(t)}).
\]

In what follows, the constant \( C > 0 \) represents different constants.

**Lemma 3.1** If \( u \) is the solution of the system (1) on \((t_0, t_1)\), then equation (6) is satisfied. Inversely, if \( u \) is a continuous function of \((t_0, t_1)\) into \( X_i^\alpha \),

\[
\int_{t_0}^{t_0+\rho} \| f(s, u(s)) \|_m \, ds < \infty
\]

for some \( \rho > 0 \) and equation (6) is satisfied for \( t_0 < t < t_1 \), then \( u \) is a solution of the system (1) on \((t_0, t_1)\).
Proof. Let assume that $u$ is the solution of the system (1) on $(t_0, t_1)$, taking $i \in [1, m]$ fixed, we define the auxiliary function
\[
g_i(t, v) = f_i(t, u_1, ..., u_{i-1}, v, u_{i+1}, ..., u_n).
\]
Let see that $g_i(t, u_i(t))$ is locally Hölder continuous in $t$ and
\[
j_0^{t_0+\rho} \|g_i(t, u_i(t))\| dt < +\infty.\]
Indeed, since $g_i : (t_0, t_1) \times X_i^\alpha \rightarrow X$ and
\[
\|g_i(t, u_i(t)) - g_i(s, u_i(s))\| = \|f_i(t, u(t)) - f_i(s, u(s))\| \leq L |t - s|^\alpha
\]
since $t \rightarrow f_i(u(t))$ is Hölder continuous with exponent $\nu \in (0, 1)$. Furthermore,
\[
\int_{t_0}^{t_0+\rho} \|g_i(t, u_i(t))\| \alpha \frac{dt}{t} = \int_{t_0}^{t_0+\rho} \|f_i(t, u(t))\| \frac{dt}{t} < +\infty
\]
for some $\rho > 0$. Now, since $u$ verifies the system (1), we have that
\[
\begin{cases}
\partial_t u_i + A_i u_i = g_i(t, u_i), \\
u_i(0) = u_{0i}.
\end{cases}
\]
Therefore by the theorem 3.2.2 in (Henry, 1981), we have that $u_i$ is the unique solution of the system (7), which can be written as
\[
u_i(t) = e^{-A_i(t-t_0)} u_{0i} + \int_{t_0}^{t} e^{-A_i(t-s)} g_i(s, u_i(s)) ds.
\]
Repeating the same procedure for all $i \in [1, m]$, we have
\[
u(t) = P(t-t_0) u_0 + \int_{t_0}^{t} P(t-s) f(s, u(s)) ds
\]
namely $u$ satisfy the equation (6).

Reciprocally, we suppose now that $u$ satisfy the equation (6) and $u \in C((t_0, t_1); X_i^\alpha)$. Besides, for each $i \in [1, m]$, we have that $u_i : (t_0, t_1) \rightarrow X_i^\alpha$ is continuous and verifies
\[
u_i(t) = e^{-A_i(t-t_0)} u_{0i} + \int_{t_0}^{t} e^{-A_i(t-s)} g_i(s, u_i(s)) ds.
\]
First, we will prove that $u_i$ is locally Hölder continuous from $(t_0, t_1)$ to $X_i^\alpha$. Thus, if $t, t + h \in [t_0, t_1] \subset (t_0, t_1)$ with $h > 0$ and $\delta_i \in (0, 1 - \alpha)$, we claim that
\[
\|u_i(t + h) - u_i(t)\|_{\alpha, i} \leq C_i h^{\delta_i}.
\]
for some positive constant $C_i$. Indeed,
\[
u_i(t + h) - u_i(t) = (e^{-A_i h} - I) e^{-A_i (t-t_0)} u_{0i} + \int_{t_0}^{t} (e^{-A_i h} - I) e^{-A_i (t-s)} g_i(s, u_i(s)) ds
\]
\[
+ \int_{t}^{t+h} e^{-A_i (t+h-s)} g_i(s, u_i(s)) ds.
\]
Now, for any $z \in X$, by Theorem 1.4.3 in (Henry, 1981),
\[
\|e^{-A_i h} - I\| e^{-A_i (t-s)} z \leq C (t - s)^{-(\alpha + \delta_i)} e^{h_{\delta_i} (t - s)} \|z\|.
\]
Moreover, due to each $f_i$ is locally Hölder in $t$ and locally Lipschitz in $u$, we have that
\[
\|f_i(t, u_i(t)) - f_i(t_0, u_i(t_0))\| \leq L_i (|t - t_0|^\theta + \|u(t) - u(t_0)\|_{\alpha})
\]
or equivalently
\[
\|g_i(t, u_i(t)) - g_i(t_0, u_i(t_0))\| \leq L_i (|t - t_0|^\theta + \|u(t)\|_{\alpha} + \|u(t_0)\|_{\alpha}).
\]
But for hypothesis, we know that $u : (t_0, t_1) \rightarrow X_i^\alpha$ is continuous, then, we have
\[
C_i = \|u(t_0)\|_{\alpha} + \max_{t_0 \leq t \leq t_1} \|u\|_{\alpha} < +\infty.
\]
Hence, by the inequality (10)
\[
\|g_i(t, u_i(t))\| \leq L (|t - t_0|^\theta + 2C_i) + \|g_i(t_0, u_i(t_0))\| \leq L (|t - t_0|^\theta + c).
\]
We begin bounding the first term of the equation (9), thus
\[
\|(e^{-A_i h} - I) e^{-A_i (t-t_0)} u_{0i}\|_{\alpha, i}
\leq C(t_0^\delta - t_0^{-(\alpha + \delta_i)} e^{h_{\delta_i} (t_0 - t_0)} \|u_{0i}\|
\leq C h^{\delta_i},
\]
since $t \in [t_0^\delta, t_1^\delta] \subset (t_0, t_1)$. Now, let us bound the second term of the equation (9)
\[
\int_{t_0}^{t} (e^{-A_i h} - I) e^{-A_i (t-s)} g_i(s, u_i(s)) ds \|_{\alpha, i}
\leq \int_{t_0}^{t} C(t - s)^{-(\alpha + \delta_i)} e^{h_{\delta_i} (t - s)} \|g_i(s, u_i(s))\| ds
\leq C h^{\delta_i} \int_{t_0}^{t} (t - s)^{-(\alpha + \delta_i)} e^{h_{\delta_i} (t - s)} (L |t - t_0|^\theta + c) ds
\leq C h^{\delta_i} \int_{t_0}^{t} (t - s)^{-(\alpha + \delta_i)} ds
Bounding now the third term of the equation (9), taking
Re \( \sigma(B_i) > \gamma_i > 0 \) and Re(\( \sigma(-a_iI) \)) \( \geq -(a_i + \gamma_i) \), using inequalities (3) and (4), we have

\[
\begin{aligned}
\| \int_{t}^{t+h} e^{-A_i(t+h-s)}g_i(s,u_i(s))ds \|_{a,i} & \leq Ch^\delta.
\end{aligned}
\]

Now, we are in position to state our main result in which we establish the existence and uniqueness of solutions to the system (1).

Theorem 3.1 If \( f_i \) verifies the hypothesis (2) for each \( i \in [1,m] \), then for any \((t_0,u_0) \in \Omega\) there exists \( T = T(t_0,u_0) > 0 \) such that the system (1) has an unique solution \( u \) on \((t_0,t_0 + T)\) with initial condition \( u(t_0) = u_0 \).

Proof. By the previous lemma is enough to find a solution \( u \) of the equation (6). We choose \( \delta > 0, \tau > 0 \) such that the set

\[
V = \{ (t,x)|t_0 \leq t \leq t_0 + \tau, \|x - u_0\|_a \leq \delta \}
\]

is contained in \( \Omega \) and

\[
\| f_i(t,x) - f_i(t,y) \| \leq L_i \| x - y \|_a \quad (11)
\]

for any \((t,x),(t,y) \in V\). Moreover, we claim that for all \( i \in [1,m] \), there exists a constant \( M_i > 0 \) such that

\[
\| B_i^a e^{-A_i t} \| \leq M_i t^{-a} e^{\alpha t} \quad (12)
\]

for all \( t > 0 \). Indeed, taking \( Re \sigma(B_i) > \gamma_i > 0 \) and \( Re(\sigma(-a_iI)) \geq -(a_i + \gamma_i) \), using inequalities (3) and (4),

\[
\| B_i^a e^{-A_i t} \| = \| B_i^a e^{-A_i t} e^{-a_i t} e^{\alpha t} \| \leq \| B_i^a e^{-B_i t} e^{\alpha t} \| \leq \| B_i^a e^{-B_i t} \| \| e^{\alpha t} \| \leq C_a^a t^{-a} e^{-\gamma t} \| e^{-a_i t} \| \leq C_a^a t^{-a} e^{-\gamma t} e^{(a_i + \gamma_i) t} \leq M_i t^{-a} e^{\alpha t}.
\]

Furthermore, we set

\[
\kappa = \max_{i \in [1,m]} \| f_i(t,u_0) \|, \quad L = \max_{i \in [1,m]} L_i
\]

and

\[
a = \max_{i \in [1,m]} a_i, \quad M = \max_{i \in [1,m]} M_i.
\]

Hence, we can choose \( T \in (0, \tau) \) such that

\[
\| (e^{-A_i h} - I) u_{0i} \|_{a,i} \leq \frac{\delta}{2m}, \quad \text{for} \ h \leq T
\]

for all \( i \in [1,m] \) and

\[
mM(\kappa + L) \int_0^T u^{-a} e^{au} du \leq \frac{\delta}{2}
\]
If $S$ denote the set of continuous functions $y : [t_0, t_0 + T] \to X^m$ such that $\|y(t) - u_0\|_a \leq \delta$ on $t_0 \leq t \leq t_0 + T$, provided by the norm

$$\|y\|^T = \sum_{i=1}^m \sup \{\|y_i(t)\|_{a,i}, \quad t_0 \leq t \leq t_0 + T\}$$

then $S$ is a complete metric space since it is the product of complete metric spaces with the product norm. So, for $y \in S$, we define $H(y) : [t_0, t_0 + T] \to X^m$ given by

$$H(y)(t) = P(t - t_0)u_0 + \int_{t_0}^t P(t - s)f(s, y(s))ds.$$

We claim that $H : S \to S$ is a contraction. Indeed, if $y \in S$ and $t_0 \leq t \leq t_0 + T$, we have that

$$\|H(y)(t) - u_0\|_a$$

\begin{align*}
\leq & \|\int_{t_0}^t P(t - s)u_0\|_a \\
+ & \int_{t_0}^t \|P(t - s)f(s, y(s))\|_a ds \\
= & \sum_{i=1}^m \left(\int_{t_0}^t \|e^{-A_i(t-s)}f_i(s, y(s))\|_{a,i} ds\right) \\
+ & \sum_{i=1}^m \int_{t_0}^t \|e^{-A_i(t-s)}f_i(s, y(s))\|_{a,i} ds
\end{align*}

\begin{align*}
\leq & \frac\alpha 2 + \frac\delta 2 + \frac{mM}{2} \int_{t_0}^t (t - s)^{-\alpha} e^{\alpha(t-s)}(L\|y(s) - u_0\|_a + \kappa) ds \\
\leq & \frac\delta 2 + m\lambda L_\delta \int_{t_0}^t u^{-\alpha} e^{\alpha(t-s)}ds \leq \delta.
\end{align*}

We prove now that $H(y) : [t_0, t_0 + T] \to X^m$ is continuous. Indeed, without loss of generality, we suppose that $z < t$, thus

$$\|H(y)(t) - H(y)(z)\|_a$$

\begin{align*}
= & \sum_{i=1}^m \|H_i(y)(t) - H_i(y)(z)\|_{a,i} \\
\leq & \sum_{i=1}^m \left(\int_{t_0}^t \|e^{-A_i(t-s)} - e^{-A_i(z-s)}\|_{a,i} ds\right) \\
+ & \int_{t_0}^t \|e^{-A_i(t-s)} - e^{-A_i(z-s)}\|_{a,i} ds \\
& \quad + \int_{z}^t \|e^{-A_i(t-s)}f_i(s, y(s))\|_{a,i} ds
\end{align*}

Let $\epsilon > 0$, we need to find $\delta > 0$ such that $|t - z| < \delta$ implies $\|H(y)(t) - H(y)(z)\|_a < \epsilon$. Thus, we bound each term of the last inequality

$$I_1 = \|e^{-A_i(t-z)} - e^{-A_i(z-t)}\|_{a,i}$$

\begin{align*}
\leq & \|B_i^\alpha e^{-A_i(z-t)} - e^{-A_i(z-t)}\|_{a,i} \\
\leq & \|B_i^\alpha\| \|e^{-A_i(z-t)}\|_{a,i} \\
\leq & \|B_i^\alpha\| c_i\epsilon e^{-b_i(z-t)}u_0 - u_0 \|_{a,i} \\
\leq & C \|e^{-A_i(z-t)}u_0 - u_0\|_a < \frac\epsilon {3m}
\end{align*}

the last inequality is satisfied if $|t - z| < \delta_1$ for some $\delta_1 > 0$.

Now, bounding $I_2$

\begin{align*}
I_2 = & \|f_i(z) - f_i(s, y(s))\|_{a,i} ds \\
\leq & \int_{t_0}^t \|B_i^\alpha (e^{-A_i(z-t)} - e^{-A_i(z-s)})\|_{a,i} ds \\
\leq & \|B_i^\alpha\| \int_{t_0}^t \|e^{-A_i(t-s)}\|_{a,i} ds \\
\leq & \|B_i^\alpha\| C \|e^{-A_i(z-t)}\|_{a,i} ds \\
\leq & C \|e^{-A_i(z-t)}\|_{a,i} ds \\
\leq & \frac\epsilon {3m}
\end{align*}

the last inequality is satisfied if $|t - z| < \delta_2$ for some $\delta_2 > 0$.

Now, proceeding with $I_3$,

\begin{align*}
I_3 = & \|f_i(z) - f_i(s, y(s))\|_{a,i} ds \\
\leq & \|B_i^\alpha\| \int_{t_0}^t \|e^{-A_i(t-s)}\|_{a,i} ds \\
\leq & C \int_{t_0}^t e^{-b_i(t-s)} ds \\
\leq & \frac\epsilon {3m}
\end{align*}

with $|t - z| < \delta_3$ for some $\delta_3 > 0$. Therefore, taking $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ we conclude that $H(y) \in C([t_0, t_0 + T]; X^m)$ and then $H(y) \in S$.

We now prove that $H$ is a contraction. Let $y, z \in S$ and $t_0 \leq t \leq t_0 + \tau$

$$\|H_i(y)(t) - H_i(z)(t)\|_{a,i}$$

\begin{align*}
\leq & \int_{t_0}^t \|B_i^\alpha\| \|e^{-A_i(t-s)}\|_{a,i} ds \\
\leq & \int_{t_0}^t M_i(t - s)^{-\alpha} e^{\alpha(t-s)} ds \|y(z) - z\|_a \\
= & ML \int_{t_0}^t (t - s)^{-\alpha} e^{\alpha(t-s)} ds \|y - z\|^T
\end{align*}
Indeed, let $C \times \Omega$.

Proof.

Firstly, we can see that if $\alpha \leq \beta < 1$ and $t \leq t_1$, then

$$
\|u(t)\|_\beta = \sum_{i=1}^m \|u_i(t)\|_{x_i^\beta}
$$

$$
\leq \sum_{i=1}^m \|e^{-A_i(t-t_0)}u_0\|_{x_i^\beta} + \int_{t_0}^t \|e^{-A_i(t-s)}f_i(s,u(s))\|_{x_i^\beta} ds
$$

$$
= \sum_{i=1}^m \left[ \|B_i^\alpha B_i^{\beta-a} e^{-A_i(t-t_0)}u_0\|_{x_i^\beta} + \int_{t_0}^t \|B_i^\beta e^{-A_i(t-s)}\|_{x_i^\beta} f_i(s,u(s))\|_{x_i^\beta} ds \right]
$$

$$
\leq \sum_{i=1}^m \left[ m C_i \|B_i^\alpha \|_{x_i^\beta} \|B_i^{\beta-a} e^{-A_i(t-t_0)}\|_{x_i^\beta} u_0\|_{x_i^\beta} \right]
$$

Thus, $\|u(t)\|_\beta$ remains bounded when $t \rightarrow t_1$. Now, we suppose that $t \leq \tau < t_1$, so

$$
u(t) - u(\tau) = (P(t-\tau) - I)u(\tau) + \int_{\tau}^t P(t-s)f(s,u(s)) ds,
$$

then

$$
\|u(t) - u(\tau)\|_{x_i^\beta} \leq \sum_{i=1}^m \left[ c \|e^{-A_i(t-\tau)}\|_{x_i^\beta} \|u_i(\tau)\|_{x_i^\beta} + \int_{\tau}^t \|e^{-A_i(t-s)}\|_{x_i^\beta} f_i(s,u(s))\|_{x_i^\beta} ds \right]
$$

$$
\leq m \sum_{i=1}^m \left[ c \|B_i^\alpha \|_{x_i^\beta} \|B_i^{\beta-a} e^{-A_i(t-\tau)}\|_{x_i^\beta} f_i(s,u(s))\|_{x_i^\beta} ds \right].
$$

Bounding the first term, let $\epsilon_1 > 0$ an arbitrary number. Since $D(A_i^{\beta-a})$ is dense in $X$, we take $v \in D(A_i^{\beta-a})$ such that $\|v\| + \|u_i(\tau) - v\| < \eta$, thus

$$
\|u(t) - u(\tau)\|_{x_i^\beta} \leq m \sum_{i=1}^m \left[ c \|B_i^\alpha \|_{x_i^\beta} \|B_i^{\beta-a} v\|_{x_i^\beta} + \int_{\tau}^t \|B_i^\beta v\|_{x_i^\beta} f_i(s,u(s))\|_{x_i^\beta} ds \right].
$$
To finish the proof, using the Theorem 3.1 and considering have that

\[ u_n(t_n) = \left( t_n, x_n \right) \]

\[ \forall n \in \mathbb{N} \]

\[ n \rightarrow 1 \]

Thus, we have

\[ u_n(t_n) = u(t_n) \]

\[ n \rightarrow 1 \]

we note that if \( z(t) = x_0 \) on \((t_0, t_1 + T(t_1))\) which contradict the maximality of \( t_1 \).

Finally, we state that under some extra conditions the unique solution is global in time.

**Theorem 3.3** Let us suppose that \( \Omega = (\tau, +\infty) \times X^a \) and \( f_i(t, x) \) satisfies hypothesis (2) for each \( i \in [1, m] \). Furthermore, there exists \( k \cdot \alpha \) a continuous function on \((\tau, +\infty)\) that verifies

\[ \| f(t, u) \|_m \leq k(1 + \| u \|_a) \]

for all \( (t, u) \in \Omega \). If \( t_0 > \tau \), \( u_0 \in X^a \), the unique solution of the system (1) with \( u(t_0) = u_0 \) exists for all \( t > t_0 \).

**Proof.** Firstly, we can note that hypothesis of the Theorem 3.2 are satisfied. Proceeding by contradiction, we take \( t_0 > \tau \) and assume that there exists an unique solution of the system (1) defined in \((t_0, t_1)\) where \( t_1 \) is maximal, so, for the last result exists a sequence \( t_n \rightarrow t_1^- \) such that \( \| u(t_n) \|_a \rightarrow +\infty \). However, since \( \beta < \alpha \) implies \( X^a < X^\beta \) for all \( i \in [1, m] \), taking \( t \in (t_0, t_1) \), by a similar procedure to the previous theorem and since \( K(\cdot) \) is continuous on \((\tau, \infty)\), i.e., bounded on \([t_0, t_1]\), we have

\[ \| u(t) \|_a \leq C \left( (t - t_0)^{-\alpha - \beta} + \int_{t_0}^{t} (t - s)^{-\alpha} k(s)(1 + \| u(s) \|_a) ds \right) \]

for the Bellman-Gronwall theorem, we can conclude that

\[ \| u(t) \|_a \leq Ce^{C(t-s)^{-\alpha} ds} \]
\[ \leq C \quad \forall t \in (t_1, t_1). \]

Which is a contradiction with the fact that \( \|u(t_n)\|_\alpha \to +\infty \) when \( t_n \to t_1^- \).

4. CONCLUSIONS

Similarly to the problem with a single equation, using the properties and estimations of sectorial operators, we state a general result concerning the existence and uniqueness of solutions to systems of equations, when the diffusion terms are given by sectorial generators, also, assuming additional hypothesis on the forcing term, a result of global existence in time is presented. The computations stated in the paper are based in the application of the Banach Fixed Point Theorem.

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Miguel Angel Sosa Yangari. Mathematician graduated from Escuela Politécnica Nacional, Ecuador. Doctor of Engineering Sciences, mention Mathematical Modeling at the Universidad de Chile, Chile. Doctor in Applied Mathematics at the University of Toulouse, France. Currently is Associate Professor, Level 3, Grade 5 at Escuela Politécnica Nacional.

Diego Salazar Israel Orellana. Mathematician graduated from Escuela Politécnica Nacional, Ecuador and graduate M.Sc. Pure Mathematics, UCE-EPN-USFQ. He is currently Ocasional Professor II at Escuela Politécnica Nacional.