Robustez de la Controlabilidad para la Ecuación de Onda Fuertemente Amortiguada Bajo la Influencia de Impulsos, Retardos y Condiciones no Locales

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Resumen: Este trabajo demuestra la siguiente conjetura: impulsos, retardos y condiciones no locales, bajo algunos supuestos, no destruyen algunas propiedades cualitativas del sistema planteado ya que son intrínsecas a él. Verificamos que la propiedad de controlabilidad es robusta bajo este tipo de perturbaciones para la ecuación de onda fuertemente amortiguada. Específicamente, demostramos que la capacidad de control interior aproximada de la ecuación de onda lineal fuertemente amortiguada no se destruye si agregamos impulsos, condiciones no locales y una perturbación no lineal con retraso en estado. Esto se hace mediante el uso de nuevas técnicas evitando teoremas de punto fijo empleado por A.E. Bashirov et al. En este caso el retardo nos ayuda a probar la capacidad de control aproximada de este sistema al retirar la solución de control a una curva fija en un corto intervalo de tiempo, y desde esta posición, podemos alcanzar una vecindad del estado final en el tiempo τ utilizando que la ecuación de onda lineal fuertemente amortiguada correspondiente es aproximadamente controlable en cualquier intervalo $\{t_0, \tau\}, 0 < t_0 < \tau$.

Palabras claves: controlabilidad aproximada interior, impulsos, ecuación de onda semilineal fuertemente amortiguada, retrasos, condiciones no locales, semigrupos fuertemente continuos.

Robustness of the controllability for the strongly damped wave equation under the influence of impulses, delays and nonlocal conditions

Abstract: This work proves the following conjecture: impulses, delays, and nonlocal conditions, under some assumptions, do not destroy some posed system qualitative properties since they are themselves intrinsic to it. we verified that the property of controllability is robust under this type of disturbances for the strongly damped wave equation. Specifically, we prove that the interior approximate controllability of linear strongly damped wave equation is not destroyed if we add impulses, nonlocal conditions and a nonlinear perturbation with delay in state. This is done by using new techniques avoiding fixed point theorems employed by A.E. Bashirov et al. In this case the delay help us to prove the approximate controllability of this system by pulling back the control solution to a fixed curve in a short time interval, and from this position, we are able to reach a neighborhood of the final state in time τ by using that the corresponding linear strongly damped wave equation is approximately controllable on any interval $\{t_0, \tau\}, 0 < t_0 < \tau$.

Keywords: interior approximate controllability, impulses, semilinear strongly damped wave equation, delays, nonlocal conditions, strongly continuous semigroups

1. Introduction

In this paper, we study the interior approximate controllability of the following strongly damped semilinear wave equations under the influence of impulses, delays and nonlocal conditions; without using fixed point Theorem

$$
\begin{aligned}
w'' + \eta(-\Delta)^{1/2}w' + \quad & \gamma(-\Delta)w = 1_\omega u(t,x) + f(t,w,w',w(t-r_1),\dots,w(t-r_m), \\
& w'(t-r_1)\dots,w'(t-r_m),u(t,x)) \text{ in } \Omega_\tau
\end{aligned} \tag{1}
$$

$$
\begin{cases}
w(t,x) = 0, \quad \text{on } \Omega_\partial, \\
w(s,x) + h_1(w(\tau_1+s,x),\dots,w(\tau_q+s,x)) = \phi_1(s,x), \\
w'(s,x) + h_2(w'(\tau_1+s,x),\dots,w'(\tau_q+s,x)) = \phi_2(s,x), \quad \text{in } \Omega_{-r} \\
w'(t_k^+,x) = w'(t_k^-,x) + I_k(t_k,w(t_k,x),w'(t_k,x),u(t_k,x)), \quad k = 1,\dots,p.
\end{cases} \tag{2}
$$

In the space $Z^{1/2} = D((-\Delta)^{1/2}) \times L^2(\Omega)$ where $w' = \frac{\partial w}{\partial t}$, $w'' = \frac{\partial^2 w}{\partial t^2}$, $\Omega \subset \mathbb{R}^N$, $N \geq 1$ is a bounded domain, γ and η are positive numbers.

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Along with Dirichlet boundary condition. where Δ denotes de Laplacian operator, Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$, $\partial\Omega$ denotes the boundary of Ω, $\Omega_\tau = (0,\tau] \times \Omega$, $\Omega_\partial = (0,\tau) \times \partial\Omega$, $\Omega_{-r} = [-r,0] \times \Omega$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the distributed control u belongs to $L_2([0,\tau];L_2(\Omega))$, $\phi_i : [-r,0] \times \Omega \longrightarrow \mathbb{R}$, $i = 1,2$, are continuous functions, $0 < r_1 < \ldots < r_m < r$ are the delays and $0 < \tau_1 < \ldots < \tau_q < \tau$.

From now on, we shall assume the following hypotheses:

H1) The functions $I_k : [0,\tau] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, $k = 1,\ldots,p$, $f : [0,\tau] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} \longrightarrow \mathbb{R}$, and $h_i : \mathbb{R}^q \longrightarrow \mathbb{R}$, $i = 1,2$ are smooth enough, such that the above problem admits maild solutions according with Leiva Hugo and Sundar P. (2017); Leiva Hugo (2018).

H2) The following estimates hold:

$$|f(t,\xi_0,\vartheta_0,\xi_1,\ldots,\xi_m,\vartheta_1,\ldots,\vartheta_m,u)| \leq \rho\left(\sqrt{\sum_{l=0}^{m}(|\xi_l|^{\rho_l} + |\vartheta_l|^{\rho_l})}\right), \quad u,\xi_l,\vartheta_l \in \mathbb{R}, \tag{3}$$

where $\rho : \mathbb{R}_+ \to [0,\infty)$ is a continuous and increasing function. In particular, ρ could be given by

$$\rho(w) = a_0 w^\beta + b_0, \quad w > 0, \beta > 0, a_0, > 0.$$

Moreover,

$$y(t_k,x) = y(t_k^+,x) = \lim_{t\to t_k^+} y(t,x), \quad y(t_k^-,x) = \lim_{t\to t_k^-} y(t,x).$$

To set this problem, we shall choose the following natural Banach space:

$$PC_{t_1..t_p}([-r,\tau];Z^{1/2}) = \{ \quad z : J = [0,\tau] \to Z^{1/2} : z \in C(J';Z^{1/2}), \exists z(t_k^+,\cdot), z(t_k^-,\cdot) \text{ and } z(t_k,\cdot) = z(t_k^+,\cdot)\},$$

$J' = [-r,\tau]\backslash\{t_1,t_2,\ldots,t_p\}$ endowed with the norm $\|z\| = \sup_{t\in[-r,\tau]} \|z(t,\cdot)\|_{Z^{1/2}}$, where $z = (w,v)^\top = (w,w_t)^\top$ and

$$\|z\|_{Z^{1/2}} = \left(\int_\Omega (\left\|(-\Delta)^{1/2}w\right\|^2 + \|v\|^2)dx\right)^{1/2}, \text{ for all } z \in Z^{1/2}.$$

Remark 1. It is clear that $PC_{t_1..t_p}([-r,\tau];Z^{1/2})$ is a closed linear subspace of the Banach space of all piecewise continuous functions $PC([-r,\tau];Z^{1/2})$ with the supreme norm, which implies that $PC_{t_1..t_p}([-r,\tau];Z^{1/2})$ is a Banach space with the same norm.

We note that the interior controllability of the following strongly damped wave equation without impulses, delays and nonlocal conditions

$$\begin{cases} w'' + \eta(-\Delta)^{1/2}w' + \gamma(-\Delta)w = 1_w u(t,x), & \text{in } (0,\tau] \times \Omega, \\ w = 0, & \text{on } (0,\tau) \times \partial\Omega, \\ w(0,x) = w_0(x) \quad w'(0,x) = w_1(x), & \text{in } \Omega \end{cases} \tag{4}$$

has been proved in Larez H., Leiva Hugo and Rebaza J. (2012), where the abstract formulation is done using fractional powered spaces, some ideas are taken from it to study this present problem. Finally, the approximate controllability of the system (1) follows from the approximate controllability of the linear system (4) in any interval of the form $[\tau - \delta, \tau]$, with $0 < \delta < \tau$, and using a new technique avoid fixed point theorems by applying in (Bashirov A.E. and Ghahramanlou N. (2013)), (Bashirov et al. (2007)), (Bashirov A.E. and Mahmudov N.I. (1999)).

There are many practical examples of impulsive control systems, a chemical reactor system, a financial system with two state variables, the amount of money in a market and the savings rate of a central bank; and the growth of a population diffusing throughout its habitat modeled by a reaction-diffusion equation. One may easily visualize situations in these examples where abrupt changes such as harvesting, disasters and instantaneous stocking may occur. These problems are modeled by impulsive differential equations, and for more information see the monographs, (Lakshmikantham V., Bainov D. D. and Simeonov P.S. (1989)) and (Samoilenko A. M. and Perestyuk N.A. (1995)).

The controllability of Impulsive Evolution Equations has been studied recently for several authors, but most them study the exact controllability only, to mention: (Chalishajar D. N. (2011)), studied the exact controllability of impulsive partial neutral functional differential equations with infinite delay and (Selvi S. and Mallika Arjunan M. (2012)) studied the exact controllability for impulsive differential systems with finite delay.

To our knowledge, there are a few works on approximate controllability of impulsive semilinear evolution equations, to mention: (Chen L. and Li G (2010)) studied the Approximate controllability of impulsive differential equations with nonlocal conditions, using measure of noncompactness and Monch's fixed point theorem, and assuming that the nonlinear term $f(t,z)$ does not depend on the control variable.

Recently, in Leiva Hugo (2014a,b); Leiva Hugo and Merentes N. (2015); Leiva Hugo (2015) the approximate controllability of semilinear evolution equations with impulses has been studied applying Rothe's Fixed Point Theorem. Also, there are many papers on evolution equations with impulses and delay or with impulses and nonlocal conditions or with local conditions and delays, where not only the controllability is studied, but also other aspects are studied, such as the existence of mild solutions, synchronization, stability, etc. To mention, we have the following references: Chalishajar D. N. (2011); Chen L. and Li G (2010); Chiu K. and Li T. (2019); Guevara C. and Leiva H. (2016); Guevara C. and Leiva H. (2017); Jiang C., Zhang F. and Li Tongxing (2018); Leiva Hugo and Rojas Raul (2016); Li Tongxing, Pintus Nicola and Viglialoro Giuseppe (2019); Liang Jin, Liu James H. and Xiao Ti-Jun (2009); Qina Haiyong et al. (2017); Quin Haiyong et al. (2017); Selvi S. and Mallika Arjunan M. (2012).

2. Abstract Formulation of the Problem.

In this section, we choose a Hilbert space where system (1) can be written as an abstract differential equation; to this end, we shall use the following notations:
Let $X = U = L^2(\Omega) = L^2(\Omega,\mathbb{R})$ and consider the linear unbounded operator $A : D(A) \subset X \longrightarrow X$ defined by

$$A\varphi = -\Delta\varphi, \text{ where } D(A) = H^2(\Omega,\mathbb{R}) \cap H^1_0(\Omega,\mathbb{R}).$$

The fractional powered spaces X^α (see details in Larez H., Leiva Hugo and Rebaza J. (2012)) are given by

$$X^\alpha = D(A^\alpha) = \left\{ x \in X : \sum_{n=1}^{\infty} \lambda_n^{2\alpha} \|E_n x\|^2 < \infty \right\},$$

endowed with the norm

$$\|x\|_\alpha = \|A^\alpha x\| = \left(\sum_{n=1}^{\infty} \lambda_n^{2\alpha} \|E_n x\|^2 \right)^{1/2},$$

where $\{E_j\}$ is a family of complete orthogonal projections in X; and for the Hilbert sapce $Z^\alpha = X^\alpha \times X$ the corresponding norm is

$$\left\| \begin{pmatrix} w \\ v \end{pmatrix} \right\|_{Z^\alpha} = \sqrt{\|w\|_\alpha^2 + \|v\|^2}.$$

Proposition 1. *Given $j \geq 1$, the operator $P_j : Z^\alpha \longrightarrow Z^\alpha$ defined by*

$$P_j = \begin{bmatrix} E_j & 0 \\ 0 & E_j \end{bmatrix} \tag{5}$$

is a continuous (bounded) orthogonal projections in the Hilbert space Z^α.
Hence, the equation (1) can be written as an abstract second order ordinary differential equation in X as

$$\begin{cases} w'' + \eta A^{1/2} w' + \gamma A w = B_\omega u + f^e(t, w(t), w'(t), w_t(-r_1), \dots, w_t(-r_m), \\ \qquad w_t'(-r_1), \dots, w_t'(-r_m), u(t)), \quad t \in (0,\tau], \quad t \neq t_k. \\ w(s) + g_1(w_{\tau_1}, \dots, w_{\tau_q})(s) = \phi_1(s) \\ w'(s) + g_2(w_{\tau_1}, \dots, w_{\tau_q})(s) = \phi_2(s), \qquad s \in [-r, 0] \\ w'(t_k^+) = w'(t_k) + I_k^e(t_k, w(t_k), w'(t_k), u(t_k)), \quad k = 1, \dots, p, \end{cases} \tag{6}$$

where

$$\begin{array}{rcl} I_k^e : [0,\tau] \times Z^{1/2} \times U & \longrightarrow & X \\ (t, w, v, u)(\cdot) & \longmapsto & I_k(t, w(\cdot), v(\cdot), u(\cdot)), \end{array}$$

$$f^e : [0,\tau] \times Z^{1/2} \times C^m([-r,0]; Z^{1/2}) \times C^m([-r,0]; Z^{1/2}) \times U \longrightarrow X$$

$$(t, w, v, \varphi_1, \dots, \varphi_m, \psi_1, \dots, \psi_m, u)(\cdot) \longmapsto \begin{array}{c} f(t, w(\cdot), v(\cdot), \varphi_1(-r_1, \cdot), \dots \varphi(-r_m, \cdot,) \\ \psi_1(-r_1, \cdot), \dots, \psi_m(-r_m, \cdot), u(\cdot)), \end{array}$$

$$\begin{array}{rcl} B_\omega : U & \longrightarrow & U \\ u(\cdot) & \longmapsto & 1_\omega u(\cdot), \end{array}$$

$$\begin{array}{rcl} g_i : C^q([-r,0]; X) & \longrightarrow & C([-r,0]; Z^{1/2}) \\ g_i(\varphi_1, \dots, \varphi_q)(s, \cdot) & \longmapsto & h_i(\varphi_1(s, \cdot), \dots, \varphi_q(s, \cdot)), \quad i = 1, 2. \end{array}$$

A change of variable $v = w'$ transforms the second order equation (6) into the following first order system of ordinary differential equations with impulses, delays and nonlocal conditions in the space $Z^{1/2}$.

$$\begin{cases} z' = \mathbb{A}z + \mathscr{B}_\omega + \mathscr{F}(t, z(t), z_t(-r_1), \dots, z_t(-r_m), u), \quad t \neq t_k, \\ z(s) + g(z_{\tau_1}, \dots, z_{\tau_q})(s) = \phi(s), \quad s \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + \mathscr{I}_k(t_k, z(t_k), u(t_k)), \quad k = 1, \dots, q, \end{cases} \tag{7}$$

where $u \in C([0,\tau], U)$, $z = (w, v)^\top$, $\phi = (\phi_1, \phi_2)^\top \in C([-r,0], X)$, z_t defined as a function from $[-r,0]$ to $Z^{1/2}$ by $z_t(s) = z(t+s), -r \leq s \leq 0$,

$$\mathbb{A} = \begin{bmatrix} 0 & I_X \\ -\gamma A & -\eta A^{1/2} \end{bmatrix}$$

is an unbounded linear operator with domain $D(\mathbb{A}) = D(A) \times D(A^{1/2})$, I_X represents the identity in X,

$$\begin{array}{rcl} \mathscr{B}_\omega \;: U & \longrightarrow & Z^{1/2} \\ u & \longmapsto & (0, B_\omega u)^\top, \end{array}$$

$$\begin{array}{rcl} \mathscr{I}_k \quad [0,\tau] \times Z^{1/2} \times U & \longrightarrow & Z^{1/2} \\ (t, z, u) & \longmapsto & (0, I_k^e(t, w, v, u))^\top, \end{array}$$

$$\mathscr{F} : [0,\tau] \times Z^{1/2} \times C^m([-r,0], Z^{1/2}) \times U \longrightarrow Z^{1/2}$$

$$(t, z, \phi^1, \dots, \phi^m, u) \longmapsto \begin{pmatrix} 0 \\ f^e(t, w, v, \phi_1^1(-r_1), \dots, \phi_1^m(-r_m), \phi_2^1(-r_1), \dots, \phi_2^m(-r_m), u) \end{pmatrix},$$

and

$$\begin{array}{rcl} g : C^q([-r,0]; X) & \longrightarrow & C([-r,0]; X) \times C([-r,0]; X) \\ g(\phi^1, \dots, \phi^q)(s, \cdot) & \longmapsto & \begin{pmatrix} g_1(\phi_1^1(s, \cdot), \dots, \phi_1^q(s, \cdot)) \\ g_2(\phi_2^1(s, \cdot), \dots, \phi_2^q(s, \cdot)) \end{pmatrix} \end{array}$$

Definition 1. **(Approximate Controllability)** The system (7) is said to be approximately controllable on $[0,\tau]$ if for every $\phi = (\phi_1, \phi_2) \in C([-r,0]; \mathscr{Z}^{1/2})$, $z^1 \in Z^{1/2}$ and $\varepsilon > 0$, there exists $u \in C([0,\tau]; L^2(\Omega))$ such that the solution $z(t)$ of (1) corresponding to u verifies:

$$z(0) + h(z_{\tau_1}, \dots, z_{\tau_q})(0) = \phi(0), \quad \text{and} \quad \left\| z(\tau) - z^1 \right\|_{Z^{1/2}} < \varepsilon.$$

The hypotheses H1) and H2), together with the continuous inclusion $X^{1/2} \subset X$, yield

Proposition 2. *The function $\mathscr{F}$ satisfies the following inequality:*

$$\|\mathscr{F}(t, z, \phi_1, .., \phi_m, u)\|_{\mathscr{Z}^{1/2}} \leq \sigma \left(\|z\|_{\mathscr{Z}^{1/2}} + \sum_{l=1}^{m} \|\phi_l(-r_l)\| \right), \tag{8}$$

where $\sigma : \mathbb{R}_+ \to [0,\infty)$ is a continuous function

It is well known that the operator $\mathbb{A}$ generates a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ in the space $Z^{1/2}$, which is also analytic. Furthermore, Lemma 2.1 in Leiva Hugo (2003) yields

Proposition 3. *The semigroup $\{T(t)\}_{t\geq 0}$ generated by the operator $\mathbb{A}$ is compact and has the following representation*

$$T(t)z = \sum_{j=1}^{\infty} e^{A_j t} P_j z, \quad z \in Z^{1/2}, \quad t \geq 0, \tag{9}$$

where $\{P_j\}_{j\geq 0}$ is a complete family of orthogonal projections in the Hilbert space $Z^{1/2}$ given by (5) and

$$A_j = R_j P_j, \quad R_j = \begin{bmatrix} 0 & 1 \\ -\gamma\lambda_j & -\eta\lambda_j^{1/2} \end{bmatrix}, \quad j \geq 1.$$

Moreover, $e^{A_j t} = e^{R_j t} P_j$, the eigenvalues of R_j are

$$\lambda = -l_j^{1/2}\left(\frac{\eta \pm \sqrt{\eta^2 - 4\gamma}}{2}\right), \quad j = 1, 2, \ldots,$$

$$A_j^* = R_j P_j, \quad R_j^* = \begin{bmatrix} 0 & -1 \\ \gamma\lambda_j & -\eta\lambda_j^{1/2} \end{bmatrix}.$$

and

$$\|T(t)\| \leq M(\eta, \gamma)e^{-\beta t}, \ t \geq 0,$$

where

$$\beta = \lambda_1^{\frac{1}{2}} \operatorname{mín}\left\{ Re\left(\frac{\eta \pm \sqrt{\eta^2 - 4\gamma}}{2}\right)\right\}$$

3. Approximate Controllability of the Linear System

In this section, we shall characterize the approximate controllability of the linear system. To this end, for all $z_0 \in Z_1$ and $u \in L^2([0,\tau];U)$ the initial value problem

$$\begin{cases} z'(t) = \mathbb{A}z(t) + \mathscr{B}_\omega u(t), \quad z \in Z^{1/2}, \\ z(t_0) = z_0, \end{cases} \tag{10}$$

admits only one mild solution given by

$$z(t) = T(t-t_0)z_0 + \int_{t_0}^{t} T(t-s)\mathscr{B}_\omega u(s)ds; \ t \in [t_0, \tau], \ 0 \leq t_0 \leq \tau. \tag{11}$$

Definition 2. For the system (10) we define the following concept: The controllability map (for $\tau > 0$) $G_{\tau\delta} : L^2([\tau-\delta,\tau];U) \longrightarrow Z^{1/2}$ is defined by

$$G_{\tau\delta}u = \int_{\tau-\delta}^{\tau} T(\tau-s)\mathscr{B}_\omega u(s)ds, \tag{12}$$

whose adjoint operator $G_{\tau\delta}^*$ is

$$G_{\tau\delta}^* : Z^{1/2} \longrightarrow L^2([\tau-\delta,\tau];U)$$

is given by

$$(G_\delta^* z)(s) = \mathscr{B}_\omega^* T(\tau)z, \ \forall s \in [0,\delta], \ \forall z \in Z^{1/2}. \tag{13}$$

The Gramian controllability operator is defined as:

$$Q_{\tau\delta} = G_{\tau\delta}G_{\tau\delta}^* = \int_{\tau-\delta}^{\tau} T(\tau-s)\mathscr{B}_\omega\mathscr{B}_\omega^* T^*(\tau-s)u(s)ds. \tag{14}$$

The following lemma holds in general for a linear bounded operator $G : \mathscr{W} \longrightarrow \mathscr{Z}$ between Hilbert spaces $\mathscr{W}$ and $\mathscr{Z}$(see Curtain R.F. and Pritchard A.J. (1978), Curtain R.F. and Zwart H.J. (1995) and Leiva Hugo, Merentes N. and Sanchez J. (2012)).

Lemma 1. *The following statements are equivalent to the approximate controllability of the linear system (10) on $[\tau-\delta,\tau]$*

1. $\overline{Rang(G_{\tau\delta})} = Z^{1/2}$.
2. $\ker(G_{\tau\delta}^*) = 0$.
3. $\langle Q_{\tau\delta}z, z\rangle > 0, z \neq 0 \, in \, Z^{1/2}$.

The following Theorem is a characterization of the approximate controllability of the system (10):

Theorem 1. *(see Bashirov et al. (2007), Bashirov A.E. and Mahmudov N.I. (1999), Curtain R.F. and Pritchard A.J. (1978),Curtain R.F. and Zwart H.J. (1995) and Leiva Hugo, Merentes N. and Sanchez J. (2012)) The system (10) is approximately controllable on $[0,\tau]$ if, and only if, any one of the following conditions hold:*

1. $\operatorname{lím}_{\alpha\to 0^+} \alpha(\alpha I + Q_{\tau\delta}^*)^{-1}z = 0$.
2. *For all $z \in Z^{1/2}$ we have $G_{\tau\delta}u_\alpha = z - \alpha(\alpha I + Q_{\tau\delta})^{-1}z$, where*
$$u_\alpha = G_{\tau\delta}^*(\alpha I + Q_{\tau\delta}^*)^{-1}z, \ \alpha \in (0,1].$$
3. *Moreover, if we consider for each $v \in L_2([\tau-\delta,\tau];U)$, the sequence of controls given by*
$$u_\alpha = G_{\tau\delta}^*(\alpha I + Q_{\tau\delta}^*)^{-1}z + (v - G_{\tau\delta}^*(\alpha I + Q_{\tau\delta}^*)^{-1}G_{\tau\delta}v), \ \alpha \in (0,1]$$
we get that:
$$G_{\tau\delta}u_\alpha = z - \alpha(\alpha I + Q_{\tau\delta}^*)^{-1}(z - G_{\tau\delta}v)$$
and
$$\operatorname{lím}_{\alpha\longrightarrow 0} G_{\tau\delta}u_\alpha = z,$$
with the error $E_{\tau\delta}z$ of this approximation is given by
$$E_{\tau\delta}z = \alpha(\alpha I + Q_{\tau\delta})^{-1}(z + G_{\tau\delta}v), \ \alpha \in (0,1].$$

Remark 2. The Theorem 1 implies that the family of linear operators $\Gamma_{\tau\delta} : Z^{1/2} \longrightarrow L^2([\tau-\delta,\tau];U)$ defined for $0 \leq \alpha \leq 1$ by

$$\Gamma_{\tau\delta}z = G_{\tau\delta}^*(\alpha I + Q_{\tau\delta}^*)^{-1}z,$$

satisfies the following relation

$$\operatorname{lím}_{\alpha\longrightarrow 0} G_{\tau\delta}\Gamma_{\tau\delta} = I$$

in the strong topology.

Since the controllability of the linear system (10) was prove in Carrasco A., Leiva H. and Sanchez J.L. (2013), on $[0,\tau]$ for all $\tau > 0$, we get the following characterization for the approximate controllability of (10).

Lemma 2. *$Q_{\tau\delta} > 0$ if, and only if, the linear system (10) is controllable on $[\tau-\delta,\tau]$. Moreover, given an initial state y_0 and a final state $Z^{1/2}$ we can find a sequence of controls $\{u_\alpha^\delta\}_{0<\alpha\leq 1} \subset L^2(\tau-\delta,\tau;U)$*

$$u_\alpha = G_{\tau\delta}^*(\alpha I + G_{\tau\delta}G_{\tau\delta}^*)^{-1}(z_1 - T(\tau)y_0), \quad \alpha \in (0,1],$$

such that the solutions $y(t) = y(t,\tau-\delta,y_0,u_\alpha^\delta)$ of the initial value problem

$$\begin{cases} y' = \mathbb{A}y + B_\omega u_\alpha(t), \ y \in Z^{1/2}, \ t > 0, \\ y(\tau-\delta) = y_0, \end{cases} \tag{15}$$

satisfies

$$\lim_{\alpha\to 0^+} y(\tau,\tau-\delta,y_0,u_\alpha)=z_1.$$

e.i.,

$$\lim_{\alpha\to 0^+} y(\tau)=\lim_{\alpha\to 0^+}\left\{T(\delta)y_0+\int_{\tau-\delta}^{\tau}T(\tau-s)Bu_\alpha(s)ds\right\}=z_1.$$

4. The System with Impulses, Delays and Nonlocal Conditions

In this section, we shall prove the main result of this paper, the interior approximate controllability of the semilinear strongly damped wave equation with impulses, delays and nonlocal conditions given by (1), which is equivalent to prove the approximate controllability of the system (7). To this end, for all $\phi\in C$ and $u\in C([0,\tau];U)$ the initial value problem, according with the recent work from Leiva Hugo and Sundar P. (2017); Leiva Hugo (2018),

$$\begin{cases} z'=\mathbb{A}z+\mathscr{B}_\omega u+\mathscr{F}(t,z(t),z_t(-r_1),z_t(-r_2),\dots,z_t(-r_m),u), & t\in(0,\tau],\\ z(s)+g(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q})(s)=\phi(s), & s\in[-r,0],\\ z(t_k^+)=z(t_k^-)+\mathscr{I}_k(t_k,z(t_k),u(t_k)), & k=1,2,3,\dots,p.\end{cases} \tag{16}$$

admits only one mild solution $z\in PC_{t_1..t_P}([-r,\tau];Z^{1/2})$ given by

$$\begin{aligned} z(t) &= T(t)\phi(0)-T(t)[(g(z_{t_1},\dots,z_{t_q}))(0)]+\int_0^t T(t-s)\mathscr{B}_\omega u(s)ds \quad (17)\\ &+ \int_0^t T(t-s)\mathscr{F}(s,z(s),z_s(-r_1),z_s(-r_2),\dots,z_s(-r_m),u(s))ds\\ &+ \sum_{0<t_k<t}T(t-t_k)\mathscr{I}_k(t_k,z(t_k),u(t_k)),\ t\in[0,\tau],\\ z(t) &+ (g(z_{\tau_1},\dots,z_{\tau_q}))(t)=\phi(t),\quad t\in[-r,0].\end{aligned}$$

Now, we are ready to present and prove the main result of this paper, which is the interior approximate controllability of the semilinear strongly damped wave equation with impulses, delays and nonlocal conditions (16).

Theorem 2. *If the functions f,I_k,h are smooth enough, condition (8) holds, and since the linear system (10) is approximately controllable on any interval $[\tau-\delta,\tau]$, $0<\delta<\tau$, then system (16) is approximately controllable on $[0,\tau]$.*

Demostración. . Given $\phi\in C$, a final state z^1 and $\varepsilon>0$, we want to find a control $u^\delta\in L^2(0,\tau;U)$ steering the system to z^1 on $[\tau-\delta,\tau]$. Precisely, for $0<\delta<min\{\tau-t_p,r\}$ small enough, there exists control $u^\delta\in L^2(0,\tau;U)$ such that corresponding of solutions z^δ of (16) satisfies

$$\left\|z^\delta(\tau)-z^1\right\|<\varepsilon.$$

In fact, we consider any fixed control $u\in L^2(0,\tau;U)$ and the corresponding solution $z(t)=z(t,0,\phi,u)$ of the problem (16). For $0<\delta<min\{\tau-t_p,r\}$ small enough, we define the control $u^\delta\in L^2(0,\tau;U)$ as follows

$$u^\delta(t)=\begin{cases} u(t), & \text{if}\quad 0\le t\le\tau-\delta,\\ v^\delta(t), & \text{if}\quad \tau-\delta<t\le\tau.\end{cases}$$

where

$$v^\delta(t)=B_\omega^*T^*(\tau-t)(\alpha I+G_{\tau\delta}G_{\tau\delta}^*)^{-1}(z^1-T(\delta)z(\tau-\delta)),\quad \tau-\delta<t\le\tau.$$

Since $0<\delta<\tau-t_p$, then $\tau-\delta>t_p$, the corresponding solution $z^\delta(t)=z(t,0,\phi,u^\delta)$ of the nonlocal Cauchy problem (16) at time τ can be written as follows:

$$\begin{aligned} z^\delta(\tau) &= T(\tau)\phi(0)-T(\tau)[(g(z_{\tau_1},\dots,z_{\tau_q})(0)]+\int_0^\tau T(\tau-s)B(s)u^\delta(s)ds\\ &+ \int_0^\tau T(\tau-s)\mathscr{F}(s,z^\delta(s),z_s^\delta(-r_1),z_s^\delta(-r_2),\dots,z_s^\delta(-r_m),u(s))\\ &+ \sum_{0<t_k<\tau}T(\tau-t_k)I_k(t_k,z(t_k),u^\delta(t_k))\\ &= T(\delta)\Big\{T(\tau-\delta)\phi(0)-T(\tau-\delta)[(g(z_{\tau_1},\dots,z_{\tau_1}))(0)]\\ &+ \int_0^{\tau-\delta}T(\tau-\delta-s)B_\omega(s)(s)u^\delta(s)ds\\ &+ \int_0^{\tau-\delta}T(\tau-\delta-s)\mathscr{F}(s,z^\delta(s),z_s^\delta(-r_1),z_s^\delta(-r_2),\dots,z_s^\delta(-r_m),u(s))ds\\ &+ \sum_{0<t_k<\tau-\delta}T(\tau-\delta-t_k)I_k^e(t_k,z^\delta(t_k),u^\delta(t_k))\Big\}+\int_{\tau-\delta}^\tau T(\tau-s)B_\omega(s)u^\delta(s)ds\\ &+ \int_{\tau-\delta}^\tau T(\tau-s)\mathscr{F}(s,z^\delta(s),z_s^\delta(-r_1),z_s^\delta(-r_2),\dots,z_s^\delta(-r_m),u(s))ds\\ &= T(\delta)z(\tau-\delta)+\int_{\tau-\delta}^\tau T(\tau-s)B_\omega(s)v^\delta(s))ds\\ &+ \int_{\tau-\delta}^\tau T(\tau-s)\mathscr{F}(s,z^\delta(s),z_s^\delta(-r_1),z_s^\delta(-r_2),\dots,z_s^\delta(-r_m),v^\delta(s))ds.\end{aligned}$$

So,

$$\begin{aligned} z^\delta(\tau) &= T(\delta)z(\tau-\delta)+\int_{\tau-\delta}^\tau T(\tau-s)B_\omega(s)v^\delta(s)ds\\ &+ \int_{\tau-\delta}^\tau T(\tau-s)\mathscr{F}(s,z^\delta(s),z_s^\delta(-r_1),z_s^\delta(-r_2),\dots,z_s^\delta(-r_m),v^\delta(s))ds.\end{aligned}$$

$$The corresponding solution$$

$y^\delta(t)=y(t,\tau-\delta,z(\tau-\delta),v^\delta)$ of the initial value problem (10) at time τ, for the control v^δ and the initial condition $z_0=z(\tau-\delta)$, is given by:

$$y^\delta(\tau)=T(\delta)z(\tau-\delta)+\int_{\tau-\delta}^\tau T(\tau-s)B_\omega(s)v^\delta(s)ds,$$

and from Lemma 2, we get a solution of the linear initial value problem (10) such that

$$\|y^\delta(\tau)-z^1\|<\frac{\varepsilon}{2}$$

Therefore,

$$\left\|z^\delta(\tau)-z_1\right\|\le\frac{\varepsilon}{2}+\int_{\tau-\delta}^\tau\|T(\tau-s)\|\left\|\mathscr{F}(s,z^\delta(s),z_s^\delta(-r_1),z_s^\delta(-r_2),\dots,z_s^\delta(-r_m),v^\delta(s))\right\|ds.$$

Now, since $0<\delta<r$ and $\tau-\delta\le s\le\tau$, then $s-r\le\tau-r<\tau-\delta$ and

$$z^\delta(s-r)=z(s-r).$$

Hence, there exists δ small enough such that $0<\delta<\text{mín}\{r,\tau-t_p\}$ and

$$\begin{aligned} \|z^\delta(\tau)-z_1\| &\le \tfrac{\varepsilon}{2}+\int_{\tau-\delta}^\tau\|T(\tau-s)\|\left\|\mathscr{F}(s,z(s),z_s(-r_1),z_s(-r_2),\dots,z_s(-r_m),v^\delta(s))\right\|ds\\ &\le \tfrac{\varepsilon}{2}+\int_{\tau-\delta}^\tau\|T(\tau-s)\|\,\sigma\left(\|z(s)\|_{\mathscr{Z}^{1/2}}+\sum_{l=1}^m\|z_s(-r_l)\|\right)\|ds<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

This completes the proof of the Theorem.

5. Final Remark

Our methodology is simple and can be applied to those second order diffusive processes with impulses, delays and nonlocal conditions like some control system governed by partial differential equations. For example, the Benjamin -Bona-Mohany Equation with impulses, delays and nonlocal conditions, the beam equations with impulses, delays and nonlocal conditions, etc.

Moreover, some of these particular problems can be formulated in a more general setting. Indeed, we can consider the following semilinear evolution equation in a general Hilbert space $Z^{1/2}$

$$\begin{cases} z' = \mathbb{A}z + \mathscr{B} + \mathscr{F}(t, z_t, u(t)), \quad t \neq t_k, \\ z(s) + g(z_{\tau_1}, \ldots, z_{\tau_q})(s) = \phi(s), \quad s \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + \mathscr{I}_k(t_k, z(t_k), u(t_k)), \quad k = 1, \ldots, q, \end{cases} \tag{18}$$

where $u \in C([0,\tau],U)$, $z = (w,v)^\top$, $\phi = (\phi_1, \phi_2)^\top \in C([-r,0],X)$, z_t defined as a function from $[-r,0]$ to $Z^{1/2}$ by $z_t(s) = z(t+s), -r \leq s \leq 0$

$$\mathbb{A} = \begin{bmatrix} 0 & I_X \\ -\gamma A & -\eta A^{1/2} \end{bmatrix}$$

is an unbounded linear operator with domain $D(\mathbb{A}) = D(A) \times D(A^{1/2})$, I_X represents the identity in X, and $A : D(A) \subset X \to Z$ is an unbounded linear operator in X with the following spectral decomposition:

$$Ax = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < x, \phi_{j,k} > \phi_{j,k},$$

with the eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \cdots \lambda_n \to \infty$ of A having finite multiplicity γ_j equal to the dimension of the corresponding eigenspaces, and $\{\phi_{j,k}\}$ is a complete orthonormal set of eigenfunctions of A. The operator $-A$ generates a strongly continuous compact semigroup $\{T_A(t)\}_{t\geq 0}$ given by

$$T_A(t)x = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} < x, \phi_{j,k} > \phi_{j,k}.$$

We shall denote by C the space of continuous functions:

$$C = \{\phi : [-r,0] \to Z^{1/2} : \phi \quad \text{is continuous}\},$$

endowed with the norm

$$\|\phi\|_C = \sup_{-r \leq s \leq 0} \|\phi(s)\|_{Z^{1/2}}.$$

The control $u \in C(0,\tau;U)$, with $U = Z$, $\mathscr{B} : X \to U$ is a linear and bounded operator(linear and continuous) and the functions $I_k^e : [0,\tau] \times Z \times U \to Z$, $\mathscr{F} : [0,\tau] \times C \times U \to Z$ satisfies the following inequalities:

$$\|\mathscr{F}(t,\phi,u)\|_{\mathscr{Z}^{1/2}} \leq \sigma(\|\phi\|_C), \tag{19}$$

where $\sigma : \mathbb{R}_+ \to [0,\infty)$ is a continuous function.
In this case the characteristic function set is a particular operator B, and the following theorem is a generalization of Lemma 2.

Theorem 3. *If vectors $\mathscr{B}^*\phi_{j,k}$ are linearly independent in $Z^{1/2}$, then the system (18) is approximately controllable on $[0,\tau]$.*

6. Conclusion

In this work, we prove the interior approximate controllability of the strongly damped equation with impulses, delays and nonlocal conditions by using a new technique avoid fixed point theorems applying by Bashirov A.E. and Ghahramanlou N. (2013), Bashirov et al. (2007), Bashirov A.E. and Mahmudov N.I. (1999). After that, we present some open problems and a possible general framework to study the controllability of semilinear second order diffusion process in Hilbert spaces with impulses, delays and nonlocal conditions. The novelty in this paper is that the literature of control systems with impulses, delays and nonlocal conditions is very short, there are a very few numbers of papers on systems with impulses, delays and nonlocal conditions simultaneously. That is to say, control systems governed by partial differential equations with impulses, delays and nonlocal conditions have not been studied much.

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REFERENCES

A.E. Bashirov and Noushin Ghahramanlou (2013), *On Partial Complete Controllability of Semilinear Systems.* Abstract and Applied Analysis, Vol. 2013, Article ID 52105, 8 pages.

A.E. Bashirov, N. Mahmudov, N. Semi and H. Etikan (2007), *On Partial Controllability Concepts.* Inernational Journal of Control, Vol. 80, No. 1, 1-7.

A.E. Bashirov and N.I. Mahmudov (1999). *On controllability of deterministic and stochastic systems.* SIAM Journal of Control and Optimization, 37, N0. 6, pp.1808-1821.

Carrasco A., Leiva H. and Sanchez J.L. (2013), *Controllability of the Semilinear Beam Equation.* J. Dyn. Control Syst., 19.

Chalishajar D. N. (2011), *Controllability of Impulsive Partial Neutral Funcional Differential Equation with Infinite Delay.* Int. Journal of Math. Analysis, Vol. 5, N0. 8, 369-380.

Chen Lizhen and Li Gang (2010), *Approximate Controllability of Impulsive Differential Equations with Nonlocal Conditions.* International Journal of Nonlinear Science, Vol.10, N0. 4, pp. 438-446.

Chiu Kuo-Shou and Li Tongxing (2019), *Oscillatory and Periodic Solutions of Differential Equations with Piecewise Constant Generalized Mixed Arguments.* Math. Nachr., in press https://doi.org/10.1002/mana.201800053.

Curtain R.F. and Pritchard A.J. (1978).*Infinite Dimensional Linear Systems.* Lecture Notes in Control and Information Sciences, 8. Springer Verlag, Berlin.

Curtain R.F. and Zwart H.J. (1995), *An Introduction to Infinite Dimensional Linear Systems Theory.* Text in Applied Mathematics, 21. Springer Verlag, New York.

Guevara C. and Leiva H. (2016), *Controllability of the Impulsive Semilinear Heat Equation with Memory and Delay.* J. Dyn Control Syst DOI 10.1007/s10883-016-9352-5.

Guevara C. and Leiva H.(2017). *Controllability of the Strongly Damped Impulsive Semilinear Wave Equation with Memory and Delay.* IMA Journal of Mathematical Control and Information, 1011. doi: 10.1093/imamci/dnx042.

Jiang C., Zhang F. and Li Tongxing (2018). *Synchronization and Antisynchronization of N-coupled Fractional-Order Complex Chaotic Systems with Ring Connection*, Math. Methos Appl. Sci., 41, 2625-2638.

Larez H., Leiva Hugo and Rebaza J. (2012). *Approximate controllability of a damped wave equation.* Canadian Applied Mathematics Quarterly, 20(3):405-419.

Leiva H., Merentes N. and Sanchez J. (2013). *Characterization of Semilinear Dense Range Operators and Applications.* Abstract and Applied Analysis. Vol. 2013. Article ID 729093, 11 pages.

Leiva Hugo (2003). *A Lemma on C_0-Semigroups and Applications PDEs Systems*, Quaestions Mathematicae, Vol. 26, pp. 247-265.

Leiva Hugo, Merentes N. and Sanchez J. (2012). *Controllability of Semilinear Reaction Diffusion*, Mathematical Control and Related Fields, Vol. 2,N0.2.

Leiva Hugo (2014). *Rothe's Fixed Point Theorem and Controllability of Semilinear Nonautonomous Systems*. System and Control Letters, 67, 14-18.

Leiva Hugo (2014). *Controllability of Semilinear Impulsive Nonautonomous Systems.* International Journal of Control, http/dx.doi.org/10.1080/00207179.2014.966759.

Leiva Hugo and Merentes N. (2015). *Approximate Controllability of the Impulsive Semilinear Heat Equation.* Journal of Mathematics and Applications, N0. 38, pp 85-104.

Leiva Hugo (2015). *Approximate Controllability of Semilinear Impulsive Evolution Equations.* Abstract and Applied Analysis, Vol. 2015, Article ID 797439, 7 pages.

Li Tongxing, Pintus Nicola and Viglialoro Giuseppe (2019). *Properties of Solutions Medium Problem with Different Sources and Boundary Conditions* , Math. Phys., 70:86.

Qina Haiyong, Zhanga Chenghui, Lia Tongxing and Chenb Ying (2017). *Controllability of Abstract Fractional Differential Evolution Equations with Nonlocal Conditions* , J. Math. Computer Sci., 17, 293-300.

Quin Haiyong, Gu Zhenyun, Fu Youliang and Tongxing Li (2017). *Existence of Mild Solutions and Controllability of Fractional Impulsive Integrodifferential Systems with Nonlocal Conditions*, J. Funct. Spaces, 2017, 1-11.

Lakshmikantham V., Bainov D. D. and Simeonov P.S. (1989). *Theory of Impulsive Differential Equations*, World Scientific, Singapore.

Leiva Hugo and Rojas Raul (2016). *Controllability of Semilinear Nonautonomous Systems with Impulses and Nonlocal Conditions*, Equilibrium, Journal of Natural Sciences, Aug. 08.

Leiva Hugo and Sundar P. (2017). *Existence of solutions for a class of semilinear evolution equations with impulses and delays*. Journal of Nonlinear Evolution Equations and Applications.

Leiva Hugo (2018). *Karakostas Fixed Point Theorem and the Existence of Solutions for Impulsive Semilinear Evolution Equations with Delays and Nonlocal Conditions*. Communications inMathematicalAnalysis, Volume 21, Number 2, pp. 68-91, ISSN 1938-9787.

Liang Jin, Liu James H. and Xiao Ti-Jun (2009). *Nonlocal Impulsive Problems for Nonlinear Differential Equations in Banach Spaces*. Mathematical and Computer Modelling 49, 798-804.

Samoilenko A. M. and Perestyuk N.A. (1995). *Impulsive Differential Equations.* World Scientific Series on Nonlinear Science Series A, Vol. 14.

Selvi S. and Mallika Arjunan M. (2012). *Controllability Results for Impulsive Differential Systems with Finite Delay*. J. Nonlinear Sci. Appl. 5, 206-219.

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