Convergence to Steady State Solutions of a Particular Class of Fractional Cooperative Systems.

Yangari M.*

*Escuela Politécnica Nacional, Departamento de Matemática, Ladrón de Guevara E11-253, Quito, Ecuador

e-mail: miguel.yangari@epn.edu.ec

Resumen: El objetivo de este artículo es probar que bajo ciertas hipótesis sobre la nolinearidad y la condición inicial, la solución de un sistema cooperativo de reacción-difusión fraccionario converge a la solución positiva más pequeña de estado estable. Además, probamos que esta convergencia es exponencial en tiempo y que el exponente de propagación depende del primer valor propio de la derivada del término de reacción y del índice más pequeño de los Laplacianos fraccionarios.

Palabras claves: Laplaciano fraccionario, ecuación nolineal de reacción-difusión, sistemas cooperativos, propagación asintótica, solución de estado estable.

Abstract: The aim of this paper is to prove that under some appropriate assumptions on the nonlinearity and the initial datum, the solution of the fractional reaction-diffusion cooperative system converge to the smallest positive steady solution. Also, we prove that this convergence is exponential in time and that the exponent of propagation depends on the principal eigenvalue of the derivative of reaction term and on the smallest index of the fractional laplacians.

Keywords: Fractional Laplacian, nonlinear reaction-diffusion equation, cooperative systems, asymptotic propagation, steady state solution.

1. INTRODUCTION

Reaction-diffusion models have found widespread applicability in a surprising number of real-world models, including areas as, chemistry, biology, physics and engineering. But not only physical phenomena can be the result of a diffusive models. Stochastic processes in mathematical finance are often modeled by a Wiener process or Brownian motion, which lead to diffusive models. The simplest reaction-diffusion models are of the form

$$u_t - \Delta u = f(u) \tag{1}$$

where f is a nonlinear function representing the reaction kinetics. One of the most important examples of particular interest for us include the Fisher-KPP equation for which $f(u) = u(1-u)$. The nontrivial dynamics of these systems arises from the competition between the reaction kinetics and diffusion.

At a microscopic level, diffusion is the result of the random motion of individual particles, and the use of Laplacian operators in the model rests on the key assumption that this random motion is an stochastic Gaussian process. However, a growing number of works have shown the presence of anomalous diffusion processes, as for example Lévy processes, thus, reaction-diffusion equations with fractional Laplacian instead of standard Laplacian appear in physical models when the diffusive phenomena are better described by Lévy processes allowing long jumps, than by Brownian processes, see for example [10] for a description of some of these models. The Lévy processes occur widely in physics, chemistry and biology and recently these models that give rise to equations with the fractional Laplacians have attracted much interest.

The reaction diffusion equation (1) with Fisher-KPP nonlinearity has been the subject of intense research since the seminal work by Kolmogorov, Petrovskii, and Piskunov [8]. Of particular interest are the results of Aronson and Weinberger [1] which describe the evolution of the compactly supported data. They showed that for a compactly supported initial value u_0, the movement of the fronts are

Submitted 24 November 2014; Accepted February 2015. The author thanks Professor J.-M. Roquejoffre for fruitful discussions.

linear in time. In addition, there exists a critical speed $c^* = 2\sqrt{f'(0)}$ for which the problem (1) admits planar traveling wave solutions connecting 0 and 1, that is, solutions of the form $u(t,x) = \phi(x - ct)$, which move with speeds $c \geq c^*$.

Moreover, Berestycki, Hamel and Roques [2] prove existence and uniqueness results for the stationary solution associated to (2) and they then analyze the behavior of the solutions of the evolution equation for large times. These results are expressed by a condition on the sign of the first eigenvalue of the associated linearized problem with periodicity condition.

In the fractional case, the anomalous diffusion problems is focussed to the study of large-time behavior of the solution of the Cauchy problem for fractional reaction-diffusion equations

$$\begin{cases} \partial_t u + (-\triangle)^\alpha u &=& f(u), \quad t > 0, x \in \mathbb{R}^d, \\ u(0,x) &=& u_0(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{2}$$

with $\alpha \in (0,1)$ in one spatial dimension, where $(-\Delta)^\alpha$ denote the fractional Laplacian. The nonlinearity f is assumed to be in the Fisher-KPP class. More precisely, the nonlinearity is assumed to have two zeros, an unstable one at $u = 0$ and a stable one at $u = 1$.

Regarding (2) with $\alpha \in (0,1)$ and Fisher-KPP nonlinearity, in connection with the discussion given above for the case $\alpha = 1$, in the recent papers [4] and [5], Cabré and Roquejoffre show that for compactly supported initial value, or more generally for initial values decaying like $|x|^{-d-2\alpha}$, where d is the dimension of the spatial variable, the speed of propagation becomes exponential in time with a critical exponent $c_* = f'(0)(d+2\alpha)^{-1}$, they also show that no traveling waves exists for this equation, all results in great contrast with the case $\alpha = 1$. Additionally we recall the earlier work in the case $\alpha \in (0,1)$ by Berestycki, Roquejoffre and Rossi [3], where it is proved that there is invasion of the unstable state by the stable one, also in [3], the authors derive a class of integro-differential reaction-diffusion equations from simple principles. They then prove an approximation result for the first eigenvalue of linear integro-differential operators of the fractional diffusion type, they also prove the convergence of solutions of fractional evolution problem to the steady state solution when the time tends to infinity.

The study of propagation fronts was also done in reaction diffusion systems, in this line, Lewis, Li and Weinberger in [9], studied spreading speeds and planar traveling waves for a particular class of cooperative reaction diffusion systems with standard diffusion by analyzing traveling waves and the convergence of initial data to wave solutions. It is shown that, for a large class of such cooperative systems, the spreading speed of the system is characterized as the slowest speed for which the system admits traveling wave solutions. Moreover, the same authors in [11] establish the existence of a explicit spreading speed σ^* for which the solution of the cooperative system spread linearly in time, when the time tends to $+\infty$.

Follow the line, when the standard Laplacians are replaced for instance by the fractional Laplacian with different indexes in a reaction diffusion cooperative systems, [6] states that the propagation speed of the solution is exponential in time with an exponent depending on the smallest index of the fractional Laplacians and of the principal eigenvalue of the matrix $DF(0)$ where F is the nonlinearity associated to the fractional system.

The aim of this paper is to prove that under some appropriate assumptions on the nonlinearity and the initial datum, the solution of the fractional reaction-diffusion cooperative systems converge to the smallest positive steady solution. Also, we prove that this convergence is exponential in time with the exponent given in [6]. More precisely, we focus on the large time behavior of the solution $u = (u_i)_{i=1}^m$, for $m \in \mathbb{N}^*$, to the fractional reaction-diffusion system:

$$\begin{cases} \partial_t u_i + (-\triangle)^{\alpha_i} u_i &=& f_i(u), \quad t > 0, x \in \mathbb{R}^d, \\ u_i(0,x) &=& u_{0i}(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{3}$$

for all $i \in [\![1,m]\!] := \{1,...,m\}$, where

$$\alpha_i \in (0,1] \quad \text{and} \quad \alpha := \min_{[\![1,m]\!]} \alpha_i < 1.$$

The operator $(-\triangle)^{\alpha_i}$ is the Fractional Laplacian defined by

$$(-\Delta)^{\alpha_i} u(x) = C(d,\alpha_i) P.V. \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|\ x - y\ |^{d+2\alpha_i}} dy$$

where the principal value is taken as the limit of the integral over $\mathbb{R}^d \setminus B_\varepsilon(x)$ as $\varepsilon \to 0$ and $C(d,\alpha)$ is a constant that depends on α_i. Note, when $\alpha_i = 1$, then $(-\triangle)^{\alpha_i} = -\triangle$. As general assumptions, we impose, for all $i \in [\![1,m]\!]$, the initial condition u_{0i} to be nonnegative, non identically equal to 0, continuous and to satisfy

$$u_{0i}(x) = O(|x|^{-(d+2\alpha_i)}) \quad \text{as} \quad |x| \to +\infty. \tag{4}$$

We also assume that for all $i \in [\![1,m]\!]$, the function $f_i \in C^1(\mathbb{R}^m)$ satisfies $f_i(0) = 0$ and that system (3) is cooperative, which means:

$$\partial_j f_i > 0, \ \text{ on } \mathbb{R}^m, \quad \text{for all } j \in [\![1,m]\!],\ j \neq i. \tag{5}$$

For what follows and without loss of generality, we suppose that $\alpha_{i+1} \leq \alpha_i$ for all $i \in [\![1,m-1]\!]$ so that $\alpha = \alpha_m < 1$. Before to continue, we need some additional hypotheses on the nonlinearity $F = (f_i)_{i=1}^m$, hence, for all $i \in [\![1,m]\!]$.

(H1) The principal eigenvalue λ_1 of the matrix $DF(0)$ is positive,

(H2) F is concave, $DF(0)$ is a symmetric matrix and $\frac{\partial f_i(0)}{\partial u_i} > 0$ for all $i \in [\![1, m]\!]$.

(H3) There exists $\Lambda > 1$ such that, for all $s = (s_i)_{i=1}^m \in \mathbb{R}_+^m$ satisfying $|s| \le \Lambda$,

$$Df_i(0)s - f_i(s) \le c_\delta \left|s_i\right|^{1+\delta},$$

(H4) For all $s = (s_i)_{i=1}^m \in \mathbb{R}_+^m$ satisfying $|s| \ge \Lambda$, we have $f_i(s) \le 0$,

(H5) $F = (f_i)_{i=1}^m$ is globally Lipschitz on $\mathbb{R}^m$,

where the constant c_δ is positive and independent of $i \in [\![1, m]\!]$ and

$$\delta \ge \frac{2}{d + 2\alpha}.$$

To state the main result, we consider ϕ the positive constant eigenvector of $DF(0)$ associated to the first eigenvalue λ_1. Thus $\lambda_1 > 0$ and $\phi > 0$ satisfy

$$\begin{aligned} (L - DF(0))\phi &= -\lambda_1 \phi \\ \phi > 0, \ \|\phi\| &= 1 \end{aligned}$$

where $L = \mathrm{diag}((-\triangle)^{\alpha_1}, ..., (-\triangle)^{\alpha_m})$. Now, let consider the autonomous system

$$\begin{aligned} \partial_t \chi_\varepsilon(t) &= F(\chi_\varepsilon(t)) \\ \chi_\varepsilon(0) &= \varepsilon\phi \end{aligned} \tag{6}$$

thus, there exists $\varepsilon' > 0$ such that, for each $\varepsilon \in (0, \varepsilon')$ we can find a constant $u_\varepsilon^+ > 0$ satisfying $\chi_\varepsilon(t) \nearrow u_\varepsilon^+$ as $t \to +\infty$, also $F(u_\varepsilon^+) = 0$. We define

$$u^+ = \inf_{\varepsilon \in (0, \varepsilon')} u_\varepsilon^+$$

since F is continuous, we deduce that $F(u^+) = 0$, therefore u^+ is a constant steady state solution of (4). Also, since the function F is positive in a small positive values close to zero, we have that $u^+ > 0$. Before to state the main result, in which we prove that the solution of (3) converge to u^+ exponentially fast in time, we assume that the initial condition u_0 satisfies

$$u_0 \le u^+ \quad \text{ in } \mathbb{R}^d \tag{7}$$

Theorem 1.1 *Let $d \ge 1$ and assume that F satisfies (5) and (H1) to (H5). Let u be the solution to (3) with u_0 satisfying (4) and (7). If $c < \frac{\lambda_1}{d+2\alpha}$, then*

$$\lim_{t \to +\infty} \inf_{|x| \le e^{ct}} |u_i(t, x) - u_i^+| = 0$$

for all $i \in [\![1, m]\!]$.

2. STEADY STATE SOLUTION

Recall that the operator $A = -\mathrm{diag}((-\Delta)^{\alpha_1}, \ldots, (-\Delta)^{\alpha_m})$ is sectorial (see [7]) in $(L^2(\mathbb{R}^d))^m$, with domain $D(A) = H^{2\alpha_1}(\mathbb{R}^d) \times \ldots \times H^{2\alpha_m}(\mathbb{R}^d)$. Thus, since $u_0 \in (L^2(\mathbb{R}^d))^m$, the Cauchy Problem (3) has a unique sectorial solution $u \in C((0, \infty), D(A)) \cap C([0, \infty), (L^2(\mathbb{R}^d))^m)$ and $\frac{du}{dt} \in C((0, \infty), (L^2(\mathbb{R}^d))^m)$.

We prove Theorem 1.1 through a sequence of lemmas. Let $B_R(0)$ be the open ball of $\mathbb{R}^d$, with center 0 and radius R, also, we denote $B_R(0)^c = \mathbb{R}^d \setminus B_R(0)$. Now, let us call u_R the unique solution of the elliptic system

$$\begin{aligned} (-\triangle)^{\alpha_i} u_i^R &= f_i(u^R), && \text{in } B_R(0) \\ u^R &= 0 && \text{on } B_R(0)^c \\ u^R &> 0 && \text{on } B_R(0) \end{aligned} \tag{8}$$

for all $i \in [\![1, m]\!]$.

Lemma 2.1 *Let $\varepsilon > 0$ and assume that F satisfies (5) and (H1) to (H3). There exists $R > 0$ such that the solution v^R of the system*

$$\begin{aligned} \partial_t v_i^R + (-\triangle)^{\alpha_i} v_i^R &= f_i(v^R), \quad t > 0, \ x \in B_R(0) \\ v^R(t, x) &= 0 \qquad \text{on } [0, \infty) \times B_R(0)^c \\ 0 < v^R(0, x) &\le \min(\varepsilon, u^R), \qquad \text{on } B_R(0). \end{aligned} \tag{9}$$

satisfies

$$\lim_{t \to +\infty} v^R(t, x) = u^R(x) \qquad \forall\, x \in B_1(0)$$

Proof: Let ϕ^R be the positive eigenvalue associated to λ_R in the ball $B_R(0)$, thus ϕ^R and λ_R satisfy

$$\begin{gathered} (L - DF(0))\phi^R = \lambda_R \phi^R \quad \text{ in } B_R(0) \\ \phi^R > 0 \ \text{ in } \ B_R(0), \ \phi^R = 0 \ \text{ in } B_R(0)^c, \ \left\|\phi^R\right\| = 1 \end{gathered}$$

Now, following the computations in [3], by (H2), we can deduce that λ_R given by the minimum of

$$\frac{\frac{1}{2}\sum_{i=1}^m \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{(\phi_i(x) - \phi_i(y))^2}{|x-y|^{d+2\alpha}} dy \right) dx}{\sum_{i=1}^m \int_{B_R(0)} \phi_i(x)^2 \, dx} - \frac{\int_{B_R(0)} [DF(0)\phi(x)] \cdot \phi(x) \, dx}{\sum_{i=1}^m \int_{B_R(0)} \phi_i(x)^2 \, dx}$$

taken over all functions $\phi \in C^1(B_R(0)) \cap C(\overline{B}_R(0))$, $\phi \not\equiv 0$, vanishing on $\partial B_R(0)$ and extended by 0 outside $B_R(0)$, where the minimum is uniquely (up to a multiplicative constant) attained by the principal eigenfunction ϕ^R, converges to $-\lambda_1$ when R goes to infinite, moreover, by hypothesis (H1) we have that $\lambda_1 > 0$, thus, we can find $R > 0$ large enough such that $\lambda_R < 0$.

Since u^R and v^R satisfy (8) and (9) in the ball $B_R(0)$, $u^R = v^R = 0$ in $B_R(0)^c$ and $v^R(0,\cdot) \le u^R(\cdot)$ in $\mathbb{R}^d$, then, by the maximum principle, we have that $v^R(t,x) \le u^R(x)$ for all $t > 0$ and $x \in \mathbb{R}^d$.

Let w^R be the solution of

$$\begin{aligned} \partial_t w_i^R + (-\triangle)^{\alpha_i} w_i^R &= f_i(w^R), \quad t>0,\ x \in B_R(0) \\ w^R(t,x) &= 0 \qquad on\ [0,\infty)\times B_R(0)^c \\ w^R(0,x) &= k\phi^R(x), \qquad on\ B_R(0). \end{aligned} \tag{10}$$

Taking $k > 0$, we deduce

$$f_i(k\phi^R) \;\ge\; kDf_i(0)\phi^R - ck^{1+\delta}(\phi_i^R)^{1+\delta}$$

Therefore, it follows from the above inequality and by the definition of ϕ^R that

$$(-\triangle)^{\alpha_i} k\phi_i^R - f_i(k\phi^R) = k\phi_i^R\left(\lambda_R + ck^\delta(\phi_i^R)^\delta\right) \le 0$$

in $B_R(0)$, for all $i \in [\![1,m]\!]$, taking k small enough and since $\lambda_R < 0$. Then $k\phi^R$ is a subsolution of (8) in the ball $B_R(0)$. Thus w^R is nondecreasing in time t. Moreover, taking $k > 0$ small if necessary, $w^R(0,x) \le v^R(0,x)$ in $\mathbb{R}^d$, thus

$$w^R(t,x) \le v^R(t,x), \qquad \forall\, t>0, x \in B_R(0)$$

Finally, one has

$$w^R(t,x) \le v^R(t,x) \le u^R(x), \qquad \forall\, t>0, x\in B_R(0)$$

Since w^R is nondecreasing in time t, standard elliptic estimates imply that w^R converges locally to a stationary solution $w^\infty(\le u^R)$ of (10). But since u^R is the unique solution of (8), we conclude that $v^R(t,x) \to u^R(x)$ in $B_R(0)$ and then, we conclude the convergence in $B_1(0)$.

Remark 2.1 *Let us note that for each $y \in \mathbb{R}^d$, if $x \in B_1(y)$ then $x - y \in B_1(0)$. Thus taking $\sigma = (\sigma)_{i=1}^m > 0$, as a consequence of Lemma 2.1, there exist $R > 0$ and $T_\sigma > 0$ that not depend of y, such that, for all $t \ge T_\sigma$*

$$|v_i^R(t,x-y) - u_i^R(x-y)| \le \sigma_i \qquad \forall\, x \in B_1(y)$$

for each $i \in [\![1,m]\!]$.

The proof of Theorem 1.1 essentially relies on the following property in which we prove that any steady state solution of (3) is bounded from below away from zero.

Lemma 2.2 *Let $d \ge 1$ and assume that F satisfies (5) and (H1) to (H3). Let v be any positive, bounded, continuous solution of*

$$(-\triangle)^{\alpha_i} v_i \;=\; f_i(v), \qquad \forall\, i \in [\![1,m]\!] \tag{11}$$

Then, there exists $\varepsilon > 0$ small enough such that $v \ge \varepsilon\phi$ in $\mathbb{R}^d$.

Proof: In what follows, we prove that there exists a constant vector $k > 0$ such that $v \ge k$ in $\mathbb{R}^d$. Let $y \in \mathbb{R}^d$ be any arbitrary fixed vector, we note that $v(\cdot + y)$ continue satisfying (11), moreover, for each $R > 0$, there exists a constant $k_{y,R} > 0$ such that $v(x+y) \ge k_{y,R}$ for all $x \in B_R(0)$.
Now, let consider the system

$$\begin{aligned} \partial_t w_i^R + (-\triangle)^{\alpha_i} w_i^R &= f_i(w^R), \quad t>0, x\in B_R(0) \\ w^R(t,x) &= 0 \qquad on\ [0,\infty)\times B_R(0)^c \\ 0 < w^R(0,x) &\le \text{mín}(k_{y,R}, u^R), \quad on\ B_R(0)^c \end{aligned} \tag{12}$$

Since $v(\cdot + y) \ge w^R(0,\cdot)$ in $\mathbb{R}^d$, by the maximum principle, we have that

$$v(x+y) \ge w^R(t,x) \qquad \forall (t,x) \in [0,\infty)\times\mathbb{R}^d \tag{13}$$

Now, by Lemma 2.1, there exists $R > 0$ large enough such that $w^R(t,x)$ converges to $u^R(x)$, as $t \to +\infty$ for all $x \in B_1(0)$. Hence, taking the limit when t tends to $+\infty$ in (13), we have that

$$v(x+y) \ge u^R(x) \qquad \forall\ x\ \in B_1(0)$$

Furthermore, taking $x = 0$ and since $y \in \mathbb{R}^d$ is arbitrary, we conclude

$$v(y) \ge u^R(0) := k \qquad \forall\ y\ \in \mathbb{R}^d$$

Finally, we take $\varepsilon > 0$ small enough such that $k \ge \varepsilon\phi$.

In the following result we state a relation between the stationary solution in the ball $B_R(0)$ and the stationary solution in the whole space.

Lemma 2.3 *Let $d \ge 1$ and assume that F satisfies (5) and (H1) to (H3). Let u^R be the solution of the system*

$$\begin{aligned} (-\triangle)^{\alpha_i} u_i^R &= f_i(u^R), \qquad \forall x \in B_R(0) \\ u^R &= 0 \qquad on\ \mathbb{R}^d \setminus B_R(0) \\ u^R &> 0 \qquad on\ B_R(0) \end{aligned} \tag{14}$$

then, u^R converges to u^+ as $R \to +\infty$, locally on compact sets.

Proof: Let $R < R'$ and $x \in B_{R'} \setminus B_R$, thus $u^R(x) = 0$, $f_i(u^R) = 0$ and $(-\triangle)^{\alpha_i} u_i^R(x) \le 0$, then we have that u^R is a subsolution of (14) on $B_{R'}$, hence, we conclude $u^R \le u^{R'}$ and therefore the sequence $\{u^R\}$ is nondecreasing in R. Moreover, since u^+ is a supersolution of (14) for all $R > 0$, we have that $u^R \le u^+$ for all radius R. Hence, the sequence $\{u^R\}$ is nondecreasing, bounded and by elliptic estimates converges in compact sets to a positive solution $v \le u^+$ of (11). Now, since $F(u^+) = 0$, then u^+ satisfies the system (11) and by Lemma 2.2, there exists $\varepsilon > 0$ such that $v \ge \varepsilon\phi$ in $\mathbb{R}^d$. Thus, we deduce that $v(x) \ge \chi_\varepsilon(t)$ for all $t \geq 0$ and $x \in \mathbb{R}^d$, where the function χ_ε satisfies (6). Therefore, taking $t \to +\infty$ and by the definition of u^+, we deduce that

$$v(x) \geq u_\varepsilon^+ \geq u^+, \qquad \forall\, x \in \mathbb{R}^d$$

Since $v \leq u^+$, we conclude that $v \equiv u^+$.

Remark 2.2 *As a consequence of Lemma 2.3, for each $\sigma = (\sigma)_{i=1}^m > 0$ and $y \in \mathbb{R}^d$, there exists $R_\sigma > 0$ that not depends of y, such that, for all $R \geq R_\sigma$*

$$|u_i^R(x-y) - u_i^+| \leq \sigma_i \qquad \forall\, x \in B_1(y)$$

for each $i \in [\![1, m]\!]$.

3. PROOF OF MAIN RESULT

Now, we can prove our main result.

Proof of Theorem 1.1: First, since $u_0(x) \leq u^+$ and u^+ satisfies the equation (3), by the maximum principle, we deduce that $u(t,x) \leq u^+$. Now, let $c < \frac{\lambda_1}{d+2\alpha}$, we take $c < c_1 < c_2 < \frac{\lambda_1}{d+2\alpha}$ fixed, thus by Theorem 1.1 of [6], there exists $\tau > 0$ and $\varepsilon = (\varepsilon_i)_{i=1}^m$, such that

$$u_i(s,x) > \varepsilon_i, \quad \text{for all } s \geq \tau \text{ and } |x| \leq e^{c_2 s} \tag{15}$$

where $u = (u_i)_{i=1}^m$ is the solution of (3).

Let $\sigma > 0$, by the Remarks 2.1 and 2.2, we can find $R_\sigma > 0$ and $T_\sigma > 0$ large enough such that for $R \geq R_\sigma$ and $s \geq T_\sigma$, we have

$$|v_i^R(s, x-y) - u_i^R(x-y)| \leq \frac{\sigma_i}{2} \tag{16}$$

and

$$|u_i^R(x-y) - u_i^+| \leq \frac{\sigma_i}{2} \tag{17}$$

for all $y \in \mathbb{R}^d$, $x \in B_1(y)$ and $i \in [\![1, m]\!]$. In what follows, taking $R \geq R_\sigma$ and τ large if necessary such that

$$R < e^{c_2\tau} - e^{c_1\tau}, \qquad e^{c_1 T_\sigma} < e^{(c_1-c)\tau}$$

we consider $y \in \{z : |z| + R \leq e^{c_2 s}\}$ with $s \geq \tau$. Then by (15), $v^R(0, \cdot - y)$ defined on $B_R(y)$ as in the Lemma 2.1 is a subsolution of (3) for times larger than s and for all $x \in \mathbb{R}^d$. Thus, by the maximum principle and (16), we have that

$$u_i(\omega + s, x) \geq u_i^R(x-y) - \frac{\sigma_i}{2}$$

for all $\omega \geq T_\sigma$ and $x \in B_1(y)$. Moreover, since $R \geq R_\sigma$ and taking $\omega = T_\sigma$, by (17)

$$u_i(s + T_\sigma, x) \geq u_i^+ - \sigma_i, \quad \text{for all } x \in B_1(y)$$

Furthermore, since $\{z : |z| \leq e^{c_1 s}\}$ is a compact set, we can find a finite number of vectors $y_1, ..., y_k$, such that $\bigcup_{i=1}^k B_1(y_i)$ cover $\{z : |z| \leq e^{c_1 s}\}$. Thus, we have

$$u_i(s + T_\sigma, x) \geq u_i^+ - \sigma_i \quad \text{for all } |x| \leq e^{c_1 s}$$

Then, taking $t = s + T_\sigma \geq \tau + T_\sigma$

$$u_i(t,x) \geq u_i^+ - \sigma_i \quad \text{for all } |x| \leq e^{-c_1 T_\sigma} e^{c_1 t}$$

thus, we conclude the proof taking $\tau_\sigma := \tau + T_\sigma$ and by election of τ, we have that

$$u_i(t,x) \geq u_i^+ - \sigma_i \quad \text{for all } |x| \leq e^{ct}.$$

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