Parabolic Systems Involving Sectorial Operators: Existence andUniqueness of Global Solutions

Miguel Yangari [1]; Diego Salazar [1]

[1] *Escuela Politécnica Nacional, Departamento de Matemática, Ladrón de Guevara E11-253, Quito, Ecuador*

Abstract: The aim of this paper is to study the existence and uniqueness of global solutions in time to systems of equations, when the diffusion terms are given by sectorial generators.

Keywords: Reaction diffusion systems, sectorial operators, infinitesimal generators.

Sistemas Parabólicos que Involucran Operadores Sectoriales: Existencia y Unicidad de Soluciones Globales

Resumen: El objetivo de este artículo es estudiar la existencia y unicidad de soluciones globales en tiempo para sistemas de ecuaciones, cuando los términos de difusión están dados por operadores sectoriales.

Palabras claves: Sistemas de reacción difusión, operadores sectoriales, operadores infinitesimales.

1. INTRODUCTION

In this paper, we study the global existence and uniqueness of sectorial solutions to the system

$$\begin{cases} \partial_t u_i + A_i u_i &= f_i(u), \quad \forall\, t \in (0,\infty) \\ u_i(0,x) &= u_{0i}(x), \end{cases}$$

where A_i are sectorial operators in the Banach space $(X, \|\cdot\|)$, $u_{0i} \in X$ and $f_i : [0,\infty) \times X^m \to X$ a given valued function for all $i \in [\![1,m]\!] := \{1,...,m\}$, where X^m is the Banach product space doted with the norm $\|u\|_m = \sum_{i=1}^m \|u_i\|$.

In the case $m = 1$, since work (Byszewski and Lakshmikantham, 1990), (Byszewski, 1991), (Byszewski, 1993), there has been increasing interest in studying abstract problems in Banach Spaces (cf., e.g., (Aizicovici and Mckibben, 2000) and references therein). For material intimately related to the present paper, we refer to (Henry, 1981), where is studied the existence of sectorial solutions for the single equations. Also in (Jackson, 1993), (Liang et al, 2002), nonlocal autonomous parabolic problems are investigated, with f being Lipschitz continuous. The existence of mild and classical solutions for reaction diffusion equations involving a particular class of sectorial operators (fractional Laplacians) are studied in (Cabrï¿œ and Roquejoffre, 2013). See also the result in (Aizicovici and Mckibben, 2000), in which $\{A(t)\}_{0\leq t\leq T}$ is a family of m-accretive operators in X generating a compact evolution family, and the existence of integral solutions to the associated nonlocal problem is shown. In the case $m > 1$, we refer to (Yangari, 2015) in which is studied the existence and uniqueness of mild solutions of a reaction diffusion system with infinitesimal generators.

In order to improve the notation, we consider the system

$$\begin{cases} \partial_t u + Au &= f(t,u), \quad \forall\, t \in (0,\infty) \\ u(0) &= u_0, \end{cases} \tag{1.1}$$

where $u = (u_i)_{i=1}^m$, $f = (f_i)_{i=1}^m$ and $A = \mathrm{diag}(A_1,...,A_m)$. Moreover, we consider the norm $\|u\|_\alpha = \sum_{i=1}^m \|u_i\|_{\alpha,i}$ on the space $X^\alpha = \prod_i^m X_i^\alpha$, where the space X_i^α is defined in the next section.

Throughout this paper, we assume Ω some open set in $\mathbb{R} \times X^\alpha$ and $f_i : \Omega \mapsto X$ is locally Hï¿œelder continuous in t and locally Lipschitz continuous in u on Ω for all $i \in [\![1,m]\!]$. More

M.Y. was supported by Proyecto Semilla PIS 14-10 of Escuela Politï¿œcnica Nacional.
Email: miguel.yangari@epn.edu.ec

precisely, if $(t_1, x_1) \in \Omega$, there exists a neighborhood $V \subset \Omega$ such that for $(t,x) \in V$, $(s,y) \in V$

$$\|f_i(t,x) - f_i(s,y)\| \le L_i(|t-s|^{\theta_i} + \|x-y\|_\alpha) \qquad (1.2)$$

for some constants $L_i > 0$ and without loss of generality we assume $\theta := \theta_i > 0$ for all $i \in [\![1,m]\!]$. Moreover, a solution of the initial value system (1.1) on (t_0, t_1) is a continuous function $u : [t_0, t_1) \longrightarrow X^m$ such that $u(t_0) = u_0$ and on (t_0, t_1) we have $(t, u(t)) \in \Omega$, $u(t) \in \mathcal{D}(A)$, $\frac{\partial u}{\partial t}(t)$ exists, $t \mapsto f(t, u(t))$ is locally Hï¿œlder continuous and

$$\int_{t_0}^{t_0+\rho} \|f(t,u(t))\|_m \, dt < \infty$$

for some $\rho > 0$ and the differential equation (1.1) is verified.

2. SECTORIAL OPERATORS

Taking $i \in [\![1,m]\!]$ fixed throughout this section, we call a linear operator A_i in a Banach space X, a sectorial operator if it is a closed densely defined operator such that, for some $\phi \in (0, \pi/2)$, $M \ge 1$ and a real ε, the sector

$$S_{\varepsilon,\phi} = \{\lambda \mid \phi \le |arg(\lambda - \varepsilon)| \le \pi, \ \ \lambda \ne \varepsilon\}$$

is in the resolvent set of A_i and

$$\left\|(\lambda I - A_i)^{-1}\right\| \le M/|\lambda - \varepsilon| \quad \text{for all } \ \lambda \in S_{\varepsilon,\phi}.$$

Let us note that every bounded linear operator on a Banach space is sectorial. Also, if we define

$$e^{-A_i t} = \frac{1}{2i\pi} \int_\Gamma (\lambda I + A_i)^{-1} e^{\lambda t} d\lambda$$

where Γ is a contour in the resolvent of $-A_i$ with $arg(\lambda) \to \pm\theta$ as $|\lambda| \to \infty$ for some $\theta \in (\pi/2, \pi)$, we have that $-A_i$ is the infinitesimal generator of the analytic semigroup $(e^{-A_i(t)})_{t\ge 0}$, moreover, if $\mathrm{Re}\,\sigma(A_i) > b_i$, then for $t > 0$

$$\left\|e^{-A_i t}\right\| \le c e^{-b_i t}, \ \ \left\|A_i e^{-A_i t}\right\| \le \frac{c}{t} e^{-b_i t}. \qquad (2.1)$$

It is important to note that, if B is a bounded linear operator, then e^{-Bt} as defined above extends to a group of linear operators and verifies

$$e^{-Bt} e^{-Bs} = e^{-B(t+s)}, \quad for -\infty < t, s < \infty.$$

In order to define the fractional power of a sectorial operator A_i, we assume $\mathrm{Re}\,\sigma(A_i) > 0$, so, for any $\alpha \in (0,1)$

$$A_i^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-A_i t} dt.$$

Taking $A_i^{-\alpha}$ defined as above, we have that this operator is a bounded linear operator on X which is one-one and satisfies $A_i^{-\alpha} A_i^{-\beta} = A_i^{-(\alpha+\beta)}$. Furthermore, A_i^α represents the inverse operator of $A_i^{-\alpha}$ with $\mathcal{D}(A_i^\alpha) = R(A_i^{-\alpha})$ and A_i^0 is the identity on X. An important result concerning positive powers of sectorial operators is

$$\left\|A_i^\alpha e^{-A_i t}\right\| \le c_\alpha t^{-\alpha} e^{-b_i t} \quad for \ \ t > 0 \qquad (2.2)$$

with $\mathrm{Re}\,\sigma(A_i) > b_i > 0$ and if $u \in \mathcal{D}(A_i^\alpha)$

$$\left\|(e^{-A_i t} - I)u\right\| \le \frac{1}{\alpha} c_{1-\alpha} t^\alpha \left\|A_i^\alpha u\right\| \qquad (2.3)$$

also, $A_i^\alpha A_i^\beta = A_i^\beta A_i^\alpha = A_i^{\alpha+\beta}$ on $\mathcal{D}(A_i^\gamma)$ with $\gamma = \max(\alpha, \beta, \alpha+\beta)$.

Now, we consider the fractional powers of $B_i := A_i + a_i I$ with $a_i \in \mathbb{R}$ chosen so $\mathrm{Re}\,\sigma(B_i) > 0$, where $\sigma(B_i)$ is the spectrum of B_i. We define the Banach space $X_i^\alpha = \mathcal{D}(B_i^\alpha)$ with the norm $\|u\|_{\alpha,i} = \left\|B_i^\alpha u\right\|$, where $\mathcal{D}(B_i^\alpha)$ is the domain of the operator B_i^α. Finally, taking $\alpha \ge \beta \ge 0$, then X_i^α is a dense subspace of X_i^β with continuous inclusion, also, $X_i^0 = X$.

For more information about sectorial operators we refer the reader to (Henry, 1981).

3. MAIN RESULTS

In order to state our first result, since $-A_i$ is the infinitesimal generator of the analytic semigroup $(e^{-A_i(t)})_{t\ge 0}$ for each $i \in [\![1,m]\!]$, we define the weak formulation for the system (1.1) given by

$$u(t) = P(t-t_0)u_0 + \int_{t_0}^t P(t-s) f(s, u(s)) ds \qquad (3.1)$$

with

$$P(t) = diag(e^{-A_1(t)}, ..., e^{-A_m(t)}).$$

In what follows, the constant $C > 0$ represents different constants.

Lemma 3.1 *If u is the solution of the system* (1.1) *on* (t_0, t_1), *then equation* (3.1) *is satisfied. Inversely, if u is a continuous function of* (t_0, t_1) *into* X^α,

$$\int_{t_0}^{t_0+\rho} \|f(s,u(s))\|_m \, ds < \infty$$

for some $\rho > 0$ and equation (3.1) *is satisfied for $t_0 < t < t_1$, then u is a solution of the system* (1.1) *on* (t_0, t_1).

Proof. Let assume that u is the solution of the system (1.1) on (t_0, t_1), taking $i \in [\![1, m]\!]$ fixed, we define the auxiliary function

$$g_i(t, v) = f_i(t, u_1, ..., u_{i-1}, v, u_{i+1}, ..., u_n).$$

Let see that $g_i(t, u_i(t))$ is locally Hï¿œlder continuous in t and $\int_{t_0}^{t_0+\rho} \|g_i(t, u_i(t))\| \, dt < +\infty$. Indeed, since $g_i : (t_0, t_1) \times X_i^\alpha \longrightarrow X$ and

$$\begin{aligned}
&\|g_i(t, u_i(t)) - g_i(s, u_i(t))\| \\
&\quad = \|f_i(t, u(t)) - f_i(s, u(s))\| \\
&\quad \le \|f(t, u(t)) - f(s, u(s))\|_m \\
&\quad \le L|t - s|^\nu
\end{aligned}$$

since $t \longrightarrow f(t, u(t))$ is Hï¿œlder continuous with exponent $\nu \in (0, 1)$. Furthermore,

$$\begin{aligned}
\int_{t_0}^{t_0+\rho} \|g_i(t, u_i(t)\|_X \, dt &= \int_{t_0}^{t_0+\rho} \|f_i(t, u(t))\|_X \, dt \\
&\le \int_{t_0}^{t_0+\rho} \|f(t, u(t))\|_m \, dt \\
&< +\infty
\end{aligned}$$

for some $\rho > 0$. Now, since u verifies the system (1.1), we have that

$$\left\{ \begin{aligned} \partial_t u_i + A_i u_i &= g_i(t, u_i), \\ u_i(0) &= u_{0i}. \end{aligned} \right. \tag{3.2}$$

Therefore by the theorem 3.2.2 in (Henry, 1981), we have that u_i is the unique solution of the system (3.2), which can be written as

$$u_i(t) = e^{-A_i(t-t_0)} u_{0i} + \int_{t_0}^t e^{-A_i(t-s)} g_i(s, u_i(s)) ds.$$

Repeating the same procedure for all $i \in [\![1, m]\!]$, we have

$$u(t) = P(t - t_0)u_0 + \int_{t_0}^t P(t - s) f(s, u(s)) ds$$

namely u satisfy the equation (3.1).

Reciprocally, we suppose now that u satisfy the equation (3.1) and $u \in \mathcal{C}\big((t_0, t_1); X^\alpha\big)$. Besides, for each $i \in [\![1, m]\!]$, we have that $u_i : (t_0, t_1) \longrightarrow X_i^\alpha$ is continuous and verifies

$$u_i(t) = e^{-A_i(t-t_0)} u_{0i} + \int_{t_0}^t e^{-A_i(t-s)} g_i(s, u_i(s)) ds.$$

First, we will prove that u_i is locally Hï¿œlder continuous from (t_0, t_1) to X_i^α. Thus, if $t, t + h \in [t_0^*, t_1^*] \subset (t_0, t_1)$ with $h > 0$ and $\delta_i \in (0, 1 - \alpha)$, we claim that

$$\|u_i(t + h) - u_i(t)\|_{\alpha,i} \le C_i h^{\delta_i} \tag{3.3}$$

for some positive constant C_i. Indeed,

$$\begin{aligned}
u_i(t + h) - u_i(t) = &(e^{-A_i h} - I)e^{-A_i(t-t_0)} u_{0i} \\
&+ \int_{t_0}^t (e^{-A_i h} - I)e^{-A_i(t-s)} g_i(s, u_i(s)) ds \\
&+ \int_t^{t+h} e^{-A_i(t+h-s)} g_i(s, u_i(s)) ds.
\end{aligned} \tag{3.4}$$

Now, for any $z \in X$, by Theorem 1.4.3 in (Henry, 1981),

$$\left\|(e^{-A_i h} - I)e^{-A_i(t-s)} z\right\|_{\alpha,i} \le C(t - s)^{-(\alpha+\delta_i)} h^{\delta_i} e^{b_i(t-s)} \|z\| .$$

Moreover, due to each f_i is locally Hï¿œlder in t and locally Lipschitz in u, we have that

$$\|f_i(t, u(t)) - f_i(t_0, u(t_0))\| \le L_i(|t - t_0|^\theta + \|u(t) - u(t_0)\|_\alpha)$$

or equivalently

$$\begin{aligned}
\|g_i(t, u_i(t)) - g_i(t_0, u_i(t_0))\| \le \; &L_i(|t - t_0|^\theta + \|u(t)\|_\alpha \\
&+ \|u(t_0)\|_\alpha).
\end{aligned} \tag{3.5}$$

But for hypothesis, we know that $u : (t_0, t_1) \longrightarrow X^\alpha$ is continuous, then, we have

$$C_\alpha = \|u(t_0)\|_\alpha + \max_{t_0^* \le t \le t_1^*} \|u\|_\alpha < +\infty.$$

Hence, by the inequality (3.5)

$$\begin{aligned}
\|g_i(t, u_i(t))\| &\le L(|t - t_0|^\theta + 2C_\alpha) + \|g_i(t_0, u_{0i})\| \\
&\le L(|t - t_0|^\theta + c).
\end{aligned}$$

We begin bounding the first term of the equation (3.4),thus

$$\begin{aligned}
&\left\|(e^{-A_i h} - I)e^{-A_i(t-t_0)} u_{0i}\right\|_{\alpha,i} \\
&\quad \le C(t_0^* - t_0)^{-(\alpha+\delta_i)} h^{\delta_i} e^{b_i(t_1-t_0)} \|u_{0i}\| \\
&\quad \le C h^{\delta_i},
\end{aligned}$$

since $t \in [t_0^*, t_1^*] \subset (t_0, t_1)$. Now, let us bound the second term of the equation (3.4)

$$\begin{aligned}
&\left\|\int_{t_0}^t (e^{-A_i h} - I)e^{-A_i(t-s)} g_i(s, u_i(s)) ds\right\|_{\alpha,i} \\
&\quad \le \int_{t_0}^t C(t - s)^{-(\alpha+\delta_i)} h^{\delta_i} e^{b_i(t-s)} \|g_i(s, u_i(s))\| \, ds \\
&\quad \le C h^{\delta_i} \int_{t_0}^t (t - s)^{-(\alpha+\delta_i)} e^{b_i(t-s)} (L|t - t_0|^\theta + c) ds \\
&\quad \le C h^{\delta_i} \int_{t_0}^t (t - s)^{-(\alpha+\delta_i)} ds
\end{aligned}$$

$$\leq \quad Ch^{\delta_i}.$$

Bounding now the third term of the equation (3.4), taking $Re\ \sigma(B_i) > \gamma_i > 0$ and $Re(\sigma(-a_iI)) \geq -(a_i+\gamma_i)$, using inequalities (2.1) and (2.2), we have

$$\begin{aligned}
&\left\| \int_t^{t+h} e^{-A_i(t+h-s)} g_i(s,u_i(s))ds \right\|_{\alpha,i} \\
&= \int_t^{t+h} \left\| B_i^{\alpha} e^{-B_i(t+h-s)} \right\| \left\| e^{a_iI(t+h-s)} g_i(s,u_i(s)) \right\| ds \\
&\leq \int_t^{t+h} C_\alpha (t+h-s)^{-\alpha} e^{a_i(t+h-s)} \left\| g_i(s,u_i(s)) \right\| ds \\
&\leq \int_t^{t+h} C_\alpha (t+h-s)^{-\alpha} e^{a_i(t+h-s)} (L|s-t_0|^{\theta}+c) ds \\
&\leq c\int_t^{t+h} (t+h-s)^{-\alpha} e^{a_i(t+h-s)} ds \\
&\leq c\int_t^{t+h} (t+h-s)^{-\alpha} e^{-\gamma t_0} e^{-\gamma t_1} e^{-\gamma h} ds \\
&\leq c\int_t^{t+h} (t+h-s)^{-\alpha} ds \\
&\leq ch^{1-\alpha} \\
&\leq Ch^{\delta_i}
\end{aligned}$$

for some large enough positive constant C. Hence, inequality (3.3) is satisfied. Moreover, $t \mapsto g_i(t,u_i(t))$ is locally Hï¿œlder continuous on (t_0,t_1). Indeed,

$$\begin{aligned}
&\|g_i(t,u_i(t)) - g_i(s,u_i(s))\| \\
&= \|f_i(t,u_i(t)) - f_i(s,u_i(s))\| \\
&\leq L_i(|t-s|^{\theta} + \|u(t)-u(s)\|_{\alpha}) \\
&\leq L_i(|t-s|^{\theta} + \sum_{i=1}^{m} \|u_i(t)-u_i(s)\|_{\alpha,i}) \\
&\leq L_i(|t-s|^{\theta} + \sum_{i=1}^{m} C_i|t-s|^{\delta_i}) \\
&\leq C(|t-s|^{\theta} + \sum_{i=1}^{m} |t-s|^{\delta_i}).
\end{aligned}$$

Also,

$$\int_{t_0}^{t_0+\rho} \|g_i(t,u_i(t))\| \leq \int_{t_0}^{t_0+\rho} \|f(t,u(t_i))\|_m dt < +\infty.$$

Then, by Theorem 3.2.2 in (Henry, 1981), u_i solves the equation

$$\begin{cases} \partial_t u_i + A_i u_i = g_i(t,u_i(t)) = f_i(t,u(t)) \\ u_i(0) = u_{0i} \end{cases}$$

for all $i \in [\![1,m]\!]$. □

Now, we are in position to state our main result in which we establish the existence and uniqueness of solutions to the system (1.1).

Theorem 3.1 *If f_i verifies the hypothesis* (1.2) *for each $i \in [\![1,m]\!]$, then for any $(t_0,u_0) \in \Omega$ there exists $T = T(t_0,u_0) > 0$ such that the system* (1.1) *has an unique solution u on (t_0,t_0+T) with initial condition $u(t_0) = u_0$.*

Proof. By the previous lemma is enough to find a solution u of the equation (3.1). We choose $\delta > 0$, $\tau > 0$ such that the set

$$V = \{(t,x) | t_0 \leq t \leq t_0+\tau, \|x-u_0\|_{\alpha} \leq \delta\}$$

is contained in Ω and

$$\|f_i(t,x) - f_i(t,y)\| \leq L_i \|x-y\|_{\alpha} \tag{3.6}$$

for any (t,x), $(t,y) \in V$. Moreover, we claim that for all $i \in [\![1,m]\!]$, there exists a constant $M_i > 0$ such that

$$\left\| B_i^{\alpha} e^{-A_i t} \right\| \leq M_i t^{-\alpha} e^{a_i t} \tag{3.7}$$

for all $t > 0$. Indeed, taking $Re\ \sigma(B_i) > \gamma_i > 0$ and $Re(\sigma(-a_iI)) \geq -(a_i+\gamma_i)$, using inequalities (2.1) and (2.2)

$$\begin{aligned}
\left\| B_i^{\alpha} e^{-A_i t} \right\| &= \left\| B_i^{\alpha} e^{-A_i t} e^{-a_i I t} e^{a_i I t} \right\| \\
&= \left\| B_i^{\alpha} e^{-B_i t} e^{a_i I t} \right\| \\
&\leq \left\| B_i^{\alpha} e^{-B_i t} \right\| \left\| e^{a_i I t} \right\| \\
&\leq C_\alpha t^{-\alpha} e^{-\gamma_i t} \left\| e^{-(-a_i I)t} \right\| \\
&\leq C_\alpha t^{-\alpha} e^{-\gamma_i t} C e^{(a_i+\gamma_i)t} \\
&\leq M_i t^{-\alpha} e^{a_i t}.
\end{aligned}$$

Furthermore, we set

$$\kappa = \max_{[t_0,t_0+\tau]} \|f_i(t,u_0)\|, \quad L = \max_{i\in[\![1,m]\!]} L_i$$

and

$$a = \max_{i\in[\![1,m]\!]} a_i, \quad M = \max_{i\in[\![1,m]\!]} M_i.$$

Hence, we can choose $T \in (0,\tau)$ such that

$$\left\| (e^{-A_i h} - I) u_{0i} \right\|_{\alpha,i} \leq \frac{\delta}{2m}, \quad for\ 0 \leq h \leq T$$

for all $i \in [\![1,m]\!]$ and

$$mM(\kappa + L\delta) \int_0^T u^{-\alpha} e^{au} du \leq \frac{\delta}{2}.$$

If S denote the set of continuous functions $y : [t_0, t_0 + T] \to X^\alpha$ such that $\|y(t) - u_0\|_\alpha \leq \delta$ on $t_0 \leq t \leq t_0 + T$, provided by the norm

$$\|y\|^T = \sum_{i=1}^{m} \sup\{\|y_i(t)\|_{\alpha,i}, \quad t_0 \leq t \leq t_0 + T\}$$

then S is a complete metric space since it is the product of complete metric spaces with the product norm. So, for $y \in S$, we define $H(y) : [t_0, t_0 + T] \to X^m$ given by

$$H(y)(t) = P(t - t_0)u_0 + \int_{t_0}^{t} P(t - s)f(s, y(s))ds.$$

We claim that $H : S \to S$ is a contraction. Indeed, if $Y \in S$ and $t_0 \leqslant t \leqslant t_0 + T$, we have that

$$\begin{aligned}
&\|H(y)(t) - u_0\|_\alpha \\
&\leq \|(P(t - t_0) - I)u_0\|_\alpha \\
&+ \int_{t_0}^{t} \|P(t - s)f(s, y(s)\|_\alpha \, ds \\
&= \sum_{i=1}^{m} \left\|(e^{-A_i(t-t_0)} - I)u_{0i}\right\|_{\alpha,i} \\
&+ \sum_{i=1}^{m} \int_{t_0}^{t} \left\|e^{-A_i(t-s)} f_i(s, y(s))\right\|_{\alpha,i} ds \\
&\leq \frac{\alpha}{2} + \sum_{i=1}^{m} \int_{t_0}^{t} \left\|(A_i)_1{}^{\alpha} e^{-A_i(t-s)}\right\| \|f_i(s, y(s)\| \, ds \\
&\leq \frac{\delta}{2} + mM \int_{t_0}^{t} (t - s)^{-\alpha} e^{a(t-s)} (L \|y(s) - u_0\|_\alpha + \kappa) ds \\
&\leq \frac{\delta}{2} + mM(\kappa + L\delta) \int_{0}^{\tau} u^{-\alpha} e^{au} du \leqslant \delta.
\end{aligned}$$

We prove now that $H(y) : [t_0, t_0 + T] \to X^\alpha$ is continuous. Indeed, without loss of generality, we suppose that $z < t$, thus

$$\begin{aligned}
&\|H(y)(t) - H(y)(z)\|_\alpha \\
&= \sum_{i=1}^{m} \|H_i(y)(t) - H_i(y)(z)\|_{\alpha,i} \\
&\leq \sum_{i=1}^{m} \Bigg[\left\|(e^{-A_i(t-t_0)} - e^{-A_i(z-t_0)})u_{0i}\right\|_{\alpha,i} \\
&\quad + \left\| \int_{t_0}^{z} (e^{-A_i(t-s)} - e^{-A_i(z-s)}) f_i(s, y(s)) ds \right\|_{\alpha,i} \\
&\quad + \left\| \int_{z}^{t} e^{-A_i(t-s)} f_i(s, y(s)) ds \right\|_{\alpha,i} \Bigg] \\
&= \sum_{i=1}^{m} (I_1 + I_2 + I_3).
\end{aligned}$$

Let $\epsilon > 0$, we need to find $\delta > 0$ such that $|t - z| < \delta$ implies $\|H(y)(t) - H(y)(z)\|_\alpha < \epsilon$. Thus, we bound each term of the last inequality

$$\begin{aligned}
I_1 &= \left\|(e^{-A_i(t-t_0)} - e^{-A_i(z-t_0)})u_{0i}\right\|_{\alpha,i} \\
&= \left\|B_i^\alpha e^{-A_i(z-t_0)} (e^{-A_i(t-z)} u_{0i} - u_{0i})\right\| \\
&\leq \|B_i^\alpha\| \left\|e^{-A_i(z-t_0)}\right\| \left\|e^{-A_i(t-z)} u_{0i} - u_{0i}\right\| \\
&\leq \|B_i^\alpha\| \, c_i e^{-b_i(z-t_0)} \left\|e^{-A_i(t-z)} u_{0i} - u_{0i}\right\| \\
&\leq C_i \left\|e^{-A_i(t-z)} u_{0i} - u_{0i}\right\| < \frac{\epsilon}{3m}
\end{aligned}$$

the last inequality is satisfied if $|t - z| < \delta_1$ for some $\delta_1 > 0$. Now, bounding I_2

$$\begin{aligned}
I_2 &= \left\| \int_{t_0}^{z} (e^{-A_i(t-s)} - e^{-A_i(z-s)}) f_i(s, y(s)) ds \right\|_{\alpha,i} \\
&\leq \int_{t_0}^{z} \left\|B_i^\alpha (e^{-A_i(t-s)} - e^{-A_i(z-s)}) f_i(s, y(s))\right\| ds \\
&\leq \|B_i^\alpha\| \int_{t_0}^{z} \left\|(e^{-A_i(t-z)} - I) e^{-A_i(z-s)} f_i(s, y(s))\right\| ds \\
&\leq \|B_i^\alpha\| \, C \int_{t_0}^{z} \left\|A_i^\alpha e^{-A_i(z-s)} f_i(s, y(s))\right\| ds(t - z) \\
&\leq C(t - z) \int_{t_0}^{z} C_\alpha (z - s)^{-\alpha} e^{-b_i(z-s)} ds \\
&\leq \frac{\epsilon}{3m}
\end{aligned}$$

the last inequality is satisfied if $|t - z| < \delta_2$ for some $\delta_2 > 0$. Now, proceeding with I_3,

$$\begin{aligned}
I_3 &= \left\| \int_{z}^{t} e^{-A_i(t-s)} f_i(s, y(s)) ds \right\|_{\alpha,i} \\
&\leq \|B_i^\alpha\| \int_{z}^{t} \left\|e^{-A_i(t-s)}\right\| \|f_i(s, y(s))\| \, ds \\
&\leq C \int_{z}^{t} e^{-b_i(t-s)} ds \\
&\leq \frac{\epsilon}{3m}
\end{aligned}$$

with $|t - z| < \delta_3$ for some $\delta_3 > 0$. Therefore, taking $\delta = \text{mín}\{\delta_1, \delta_2, \delta_3\}$ we conclude that $H(y) \in \mathcal{C}([t_0, t_0 + T]; X^\alpha)$ and then $H(y) \in S$.

We now prove that H is a contraction. Let $y, z \in S$ and $t_0 \leq t \leq t_0 + \tau$

$$\begin{aligned}
&\|H_i(y)(t) - H_i(z)(t)\|_{\alpha,i} \\
&\leq \int_{t_0}^{t} \left\|B_i^\alpha e^{-A_i(t-s)}\right\| \|f_i(s, y(s)) - f_i(s, z(s))\| \, ds \\
&\leq \int_{t_0}^{t} M_i (t - s)^{-\alpha} e^{a_i(t-s)} L_i \|y(s) - z(s)\|_\alpha \\
&= ML \int_{t_0}^{t} (t - s)^{-\alpha} e^{a_i(t-s)} ds \, \|y - z\|^T
\end{aligned}$$

$$\leq \quad ML\int_0^T u^{-\alpha}e^{au}ds\,\|y-z\|^T$$

therefore, with $I=[t_0,t_0+\tau]$

$$\sup_{t\in I}\|H_i(y)(t)-H_i(z)(t)\|_{\alpha,i}\leq ML\int_0^T u^{-\alpha}e^{au}ds\,\|y-z\|^T$$

for all $i\in[\![1,m]\!]$. Hence

$$\begin{aligned}\|H(y)(t)-H(z)(t)\|^T &\leq mML\int_0^T u^{-\alpha}e^{au}ds\,\|y-z\|^T\\ &\leq \frac{1}{2}\|y-z\|^T.\end{aligned}$$

For the fixed point theorem, H has an unique solution $u\in S$, where u is a continuous function $u:[t_0,t_0+T]\to(x_i^\alpha)^m$ which verifies the equation (3.1). Also, f_i satisfies

$$\|f_i(t,u)-f_i(t_0,u_0)\|\leq L_i(|t-t_0|^\theta+\|u-u_0\|_\alpha)$$

thus,

$$\begin{aligned}\|f_i(t,u)\| &\leq L_i(|t-t_0|^\theta+\delta)+\|f_i(t_0,u_0)\|\\ &\leq L(T^\theta+\delta)+B\leq C\end{aligned}$$

and then, for any fixed $\rho>0$

$$\begin{aligned}\int_{t_0}^{t_0+\rho}\|f(s,u(s))\|_m ds &= \sum_{i=1}^m\int_{t_0}^{t_0+\rho}\|f_i(s,u(s))\|ds\\ &\leq mC_2\rho<+\infty.\end{aligned}$$

Therefore, for the Lemma 3.1, u is solution of the system (1.1) on (t_0,t_0+T). □

As a consequence of the previous theorem, we present a result about the behavior of the solution.

Theorem 3.2 *If f_i verifies the hypothesis* (1.2) *for each $i\in[\![1,m]\!]$ and for all closed and bounded subset $V\subset\Omega$ the image of $f(V)$ is bounded in X^m. Then u is a solution of the system* (1.1) *on (t_0,t_1) and t_1 is maximal, i.e., there is no solution of the system* (1.1) *on (t_0,t_2) if $t_2>t_1$, then either $t_1=+\infty$ or else there exists a sequence $t_n\to t_{1^-}$ as $n\to+\infty$ such that $(t_n,u(t_n))\to\partial\Omega$. If Ω is unbounded, the point at infinity is included in $\partial\Omega$.*

Proof. Proceeding by contradiction, we suppose that $t_1<+\infty$, but $(t,u(t)$ is not in a neighborhood N of $\partial\Omega$ for $\underline{t}\leq t<t_1$, we can take N of the form $N=\Omega\backslash B$, where B is a closed and bounded set of Ω and $(t,u(t))\in B$ for all $\underline{t}\leq t<t_1$. We will prove that there exists $x_1\in X^\alpha$ such that $(t_1,x_1)\in B$ with $u(t)\to x_1$ in X^α when $t\to t_1^-$. Even more $u(t_1)=x_1$, which means that the solution could be extended until t_1.
Indeed, let $C=\sup\{\|f(t,u)\|_m,(t,u)\in B\}$, so $C<+\infty$ because B is bounded and closed and by hypothesis $f(B)$ is bounded in X^m.

Firstly, we can see that if $\alpha\leq\beta<1$ and $\underline{t}\leq t<t_1$, then

$$\begin{aligned}\|u(t)\|_\beta &= \sum_{i=1}^m\|u_i(t)\|_{x_i^\beta}\\
&\leq \sum_{i=1}^m\Big[\left\|e^{-A_i(t-t_0)}u_{0i}\right\|_{\beta,i}\\
&\qquad+\int_{t_0}^t\left\|e^{-A_i(t-s)}f_i(s,u(s))\right\|_{\beta,i}ds\Big]\\
&= \sum_{i=1}^m\Big[\left\|B_i^\alpha B_i^{\beta-\alpha}e^{-A_i(t-t_0)}u_{0i}\right\|\\
&\qquad+\int_{t_0}^t\left\|B_i^\beta e^{-A_i(t-s)}\right\|\|f_i(s,u(s))\|ds\Big]\\
&\leq \sum_{i=1}^m c\|B_i^\alpha\|\left\|B_i^{\beta-\alpha}e^{-A_i(t-t_0)}\right\|\|u_{0i}\|_{\alpha,i}\\
&\qquad+\int_{t_0}^t\left\|B_i^\beta e^{-A_i(t-s)}\right\|\|f_i(s,u(s))\|ds\\
&\leq c\sum_{i=1}^m\Big[M_i(t-t_0)^{-(\beta-\alpha)}e^{a_i(t-t_0)}\|u_{0i}\|_{\alpha,i}\\
&\qquad\int_{t_0}^t M_i(t-s)^{-\beta}e^{a_i(t-s)}\|f_i(s,u(s))\|ds\Big]\\
&\leq c\Big[(t-t_0)^{-(\beta-\alpha)}+\int_{t_0}^t(t-s)^{-\beta}ds\Big]\\
&\leq C.\end{aligned}$$

Thus, $\|u(t)\|_\beta$ remains bounded when $t\to t_1^-$. Now, we suppose that $\underline{t}\leq\tau<t<t_1$, so

$$u(t)-u(\tau)=(P(t-\tau)-I)u(\tau)+\int_\tau^t P(t-s)f(s,u(s))ds,$$

then

$$\begin{aligned}&\|u(t)-u(\tau)\|_{\alpha,i}\\
&\quad\leq \sum_{i=1}^m\Big[c\left\|(e^{-A_i(t-\tau)}-I)u_i(\tau)\right\|_{\beta,i}\\
&\qquad+\int_\tau^t\left\|e^{-A_i(t-s)}f_i(s,u(s))\right\|_{\alpha,i}ds\Big]\\
&\quad\leq \sum_{i=1}^m\Big[c\|B_i^\alpha\|\left\|B_i^{\beta-\alpha}(e^{-A_i(t-\tau)}-I)u_i(\tau)\right\|\\
&\qquad+\int_\tau^t\left\|B_i^\alpha e^{-A_i(t-s)}f_i(s,u(s))\right\|ds\Big].\end{aligned}$$

Bounding the first term, let $\epsilon_1>0$ an arbitrary number. Since $\mathcal{D}(A_i^{\beta-\alpha})$ is dense in X, we take $v\in\mathcal{D}(A_i^{\beta-\alpha})$ such that $\|u_i(\tau)-v\|<\eta$, thus

$$\begin{aligned}
&\left\| B_i^{\beta-\alpha}(e^{-A_i(t-\tau)} - I)u_i(\tau) \right\| \\
&\quad \leq \left\| B_i^{\beta-\alpha} \right\| \left\| (e^{-A_i(t-\tau)} - I)u_i(\tau) \right\| \\
&\quad \leq c \left\| (e^{-A_i(t-\tau)} - I)(u_i(\tau) - v) + (e^{-A_i(t-\tau)} - I)v \right\| \\
&\quad \leq c \Big[\left\| e^{-A_i(t-\tau)}(u_i(\tau) - v) \right\| + \|u_i(\tau) - v\| \\
&\qquad\quad + \left\| (e^{-A_i(t-\tau)} - I)v \right\| \Big] \\
&\quad \leq c \Big[e^{-b_i(t-\tau)} \|u_i(\tau) - v\| + \|u_i(\tau) - v\| \\
&\qquad\quad + \left\| (e^{-A_i(t-\tau)} - I)v \right\| \Big] \\
&\quad \leq c \left[e^{-b_i t_1}\eta + \eta + (t-\tau)^{\beta-\alpha} \left\| A_i^{\beta-\alpha} v \right\| \right] \\
&\quad \leq \epsilon_1 + C(t-\tau)^{\beta-\alpha}.
\end{aligned}$$

When the last bound is given when η is small enough such that $c(e^{-a_i(t-\tau)}\eta + \eta) < \epsilon_1$. Now, bounding the second term

$$\begin{aligned}
&\int_\tau^t \left\| B_i^\alpha e^{-A_i(t-s)} f_i(s, u(s)) \right\| ds \\
&\qquad \leq c \int_\tau^t \left\| B_i^\alpha e^{-A_i(t-s)} \right\| \|f_i(s, u(s))\| \, ds \\
&\qquad \leq c M_i \int_\tau^t (t-s)^{-\alpha} e^{a_i(t-s)} ds \\
&\qquad \leq C(t-s)^{1-\alpha}.
\end{aligned}$$

Thus, we have

$$\|u(t) - u(\tau)\|_\alpha \leq C \left[\epsilon_1 + (t-\tau)^{\beta-\alpha} + (t-\tau)^{1-\alpha} \right].$$

Now, we consider $t_n \to t_1^-$ and let define $u_n = u(t_n)$, thus, taking $\epsilon > 0$, we have

$$\begin{aligned}
\|u_n - u_m\|_\alpha &\leq C \left[\epsilon_1 + (t_n - t_m)^{\beta-\alpha} + (t_n - t_m)^{1-\alpha} \right] \\
&< \epsilon.
\end{aligned}$$

if n, m are large enough. Therefore, (u_n) is a Cauchy sequence in the complete space X^α, thus, there exists $x_1 \in X^\alpha$ such that $u_n \to x_1$, it means $(t_n, u(t_n)) \to (t_1, x_1)$ and since $(t_n, u(t_n)) \in B$ with B closed, we can conclude that $(t_1, x_1) \in B$.

Also, since $u : (t_0, t_1) \to X^m$ is continuous and $t_n \to t_1^-$, we have that $(t_n, u(t_n)) \to (t_1, u(t_1)) \in B$, thus, we can conclude that $u(t_1) = x_1$.

To finish the proof, using the Theorem 3.1 and considering $(t_1, x_1) \in B \subset \Omega$, we can find an unique solution v on $(t_1, t_1 + T(t_1))$ for some $T(t_1) > 0$ of the system (1.1) with initial condition $v(t_1) = x_1$. Hence, taking

$$z(t) = \begin{cases} u(t) & if\ t \in [t_0, t_1] \\ v(t) & if\ t \in [t_1, t_1 + T(t_1)) \end{cases}$$

we note that z is continuous in $[t_0, t_1 + T(t_1))$. So, we conclude that z is a solution of the system (1.1) with $z(t_0) = x_0$ on $(t_0, t_1 + T(t_1))$ which contradict the maximality of t_1. □

Finally, we state that under some extra conditions the unique solution is global in time.

Theorem 3.3 *Let us suppose that $\Omega = (\tau, +\infty) \times X^\alpha$ and $f_i(t, x)$ satisfies hypothesis (1.2) for each $i \in [\![1, m]\!]$. Furthermore, there exists $k(\cdot)$ a continuous function on $(\tau, +\infty)$ that verifies*

$$\|f(t, u)\|_m \leq k(t)(1 + \|u\|_\alpha)$$

for all $(t, u) \in \Omega$. If $t_0 > \tau$, $u_0 \in X^\alpha$, the unique solution of the system (1.1) with $u(t_0) = u_0$ exists for all $t > t_0$.

Proof. Firstly, we can note that hypothesis of the Theorem 3.2 are satisfied. Proceeding by contradiction, we take $t_0 > \tau$ and assume that there exists an unique solution of the system (1.1) defined in (t_0, t_1) where t_1 is maximal, so, for the last result exists a sequence $t_n \to t_1^-$ such that $\|u(t_n)\|_\alpha \to +\infty$. However, since $\beta < \alpha$ implies $X_i^\alpha < X_i^\beta$ for all $i \in [\![1, m]\!]$, taking $t \in (\underline{t}, t_1)$, by a similar procedure to the previous theorem and since $K(\cdot)$ is continuous on (τ, ∞), i.e., bounded on $[t_0, t_1]$, we have

$$\begin{aligned}
\|u(t)\|_\alpha &\leq C \Big[(t-t_0)^{-(\alpha-\beta)} \\
&\qquad + \int_{t_0}^t (t-s)^{-\alpha} \|f(s, u(s))\|_m \, ds \Big] \\
&\leq c \Big[(t-t_0)^{-(\alpha-\beta)} \\
&\qquad + \int_{t_0}^t (t-s)^{-\alpha} k(s)(1 + \|u(s)\|_\alpha \, ds \Big] \\
&\leq c \Big[(t-t_0)^{-(\alpha-\beta)} \\
&\qquad + \int_{t_0}^t (t-s)^{-\alpha} (1 + \|u(s)\|_\alpha \, ds \Big] \\
&\leq c + \int_{t_0}^t c(t-s)^{-\alpha} \|u(s)\|_\alpha \, ds
\end{aligned}$$

for the Bellman-Gronwall theorem, we can conclude that

$$\|u(t)\|_\alpha \leq Ce^{\int_{t_0}^t c(t-s)^{-\alpha} ds}$$

$$\leq \quad C \qquad\qquad \forall t \in (\underline{t}, t_1).$$

Which is a contradiction with the fact that $\|u(t_n)\|_\alpha \to +\infty$ when $t_n \to t_1^-$. □

4. CONCLUSIONS

Similarly to the problem with a single equation, using the properties and estimations of sectorial operators, we state a general result concerning the existence and uniqueness of solutions to systems of equations, when the diffusion terms are given by sectorial generators, also, assuming additional hypothesis on the forcing term, a result of global existence in time is presented. The computations stated in the paper are based in the application of the Banach Fixed Point Theorem.

REFERENCES

Aizicovici S., and Mckibben M. (2000). Existence results for a class of abstract nonlocal Cauchy problems. Nonlinear Analysis. 39, 649-668.

Benchohra, M and Ntouyas, S. (2001). Nonlocal Cauchy problems for neutral functional differential and integrodifferential inclusions in Banach spaces. J. Math. Anal. Appl. 258, 573-590.

Byszewski L. and Lakshmikantham V. (1990). Theorem about the existence and uniqueness of a solutions of a nonlocal Cauchuy problem in a Banach space. Appl. Anal. 40, 11-19.

Byszewski, L. (1991). Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. J. Math. Anal. Appl. 162, 494-505.

Byszewski, L. (1993). Existence and uniqueness of solutions of semilinear evolution nonlocal Cauchy problem. Zesz. Nauk. Pol. Rzes. Mat. Fiz. 18, 109-112.

Byszewski, L. and Akca, H. (1998). Existence of solutions of a semilinear functional-differential evolution nonlocal problem. Nonlinear Analysis. 34, 65-72.

Cabré, X. and Roquejoffre, J. (2013). The influence of fractional diffusion in Fisher-KPP equation. Commun. Math. Phys. 320, 679-722.

Fu, X. and Ezzinbi, K. (2003). Existence of solutions for neutral functional differential evolution equations with nonlocal conditions. Nonlinear Analysis. 54, 215-227.

Henry, D. (1981). Geometric theory of semilinear parabolic equations. Berlin, Germany: Springer-Verlag.

Jackson, D. (1993). Existence and uniqueness of solutions to semilinear nonlocal parabolic equations. J. Math. Anal. Appl. 172, 256-265.

Liang, J., Van Casteren, J., and Xiao, T. (2002). Nonlocal Cauchy problems for semilinear evolution equations. Nonlinear Anal. Ser. A: Theory Methods. 50, 173-189.

Lin,Y. and Liu, J. (1996). Semilinear integrodifferential equations with nonlocal Cauchy Problems. Nonlinear Analysis. 26, 1023-1033.

Ntouyas, S. and Tsamotas, P. (1997). Global existence for semilinear evolution equations with nonlocal conditions. J. Math. Anal. Appl. 210, 679-687.

Ntouyas, S and Tsamotas, P. (1997). Global existence for semilinear integrodifferential equations with delay and nonlocal conditions. J. Math. Anal. Appl. 64, 99-105

Yangari, M. (2015). Existence and Uniqueness of Global Mild Solutions for Nonlocal Cauchy Systems in Banach Spaces. Revista Politï¿œcnica. 35(2), 149-152.