

# Parabolic Systems Involving Sectorial Operators: Existence and Uniqueness of Global Solutions

Miguel Yangari<sup>1</sup>; Diego Salazar<sup>1</sup>

<sup>1</sup>Escuela Politécnica Nacional, Departamento de Matemática, Ladrón de Guevara E11-253, Quito, Ecuador

**Abstract:** The aim of this paper is to study the existence and uniqueness of global solutions in time to systems of equations, when the diffusion terms are given by sectorial generators.

**Keywords:** Reaction diffusion systems, sectorial operators, infinitesimal generators.

## Sistemas Parabólicos que Involucran Operadores Sectoriales: Existencia y Unicidad de Soluciones Globales

**Resumen:** El objetivo de este artículo es estudiar la existencia y unicidad de soluciones globales en tiempo para sistemas de ecuaciones, cuando los términos de difusión están dados por operadores sectoriales.

**Palabras clave:** Sistemas de reacción difusión, operadores sectoriales, operadores infinitesimales.

### 1. INTRODUCTION

In this paper, we study the global existence and uniqueness of sectorial solutions to the system

$$\begin{cases} \partial_t u_i + A_i u_i &= f_i(u), \quad \forall t \in (0, \infty) \\ u_i(0, x) &= u_{0i}(x), \end{cases}$$

where  $A_i$  are sectorial operators in the Banach space  $(X, \|\cdot\|)$ ,  $u_{0i} \in X$  and  $f_i : [0, \infty) \times X^m \rightarrow X$  a given valued function for all  $i \in \llbracket 1, m \rrbracket := \{1, \dots, m\}$ , where  $X^m$  is the Banach product space dotted with the norm  $\|u\|_m = \sum_{i=1}^m \|u_i\|$ .

In the case  $m = 1$ , since work (Byszewski and Lakshmikantham, 1990), (Byszewski, 1991), (Byszewski, 1993), there has been increasing interest in studying abstract problems in Banach Spaces (cf., e.g., (Aizicovici and McKibben, 2000) and references therein). For material intimately related to the present paper, we refer to (Henry, 1981), where is studied the existence of sectorial solutions for the single equations. Also in (Jackson, 1993), (Liang et al, 2002), nonlocal autonomous parabolic

problems are investigated, with  $f$  being Lipschitz continuous. The existence of mild and classical solutions for reaction diffusion equations involving a particular class of sectorial operators (fractional Laplacians) are studied in (Cabr e and Roquejoffre, 2013). See also the result in (Aizicovici and McKibben, 2000), in which  $\{A(t)\}_{0 \leq t \leq T}$  is a family of  $m$ -accretive operators in  $X$  generating a compact evolution family, and the existence of integral solutions to the associated nonlocal problem is shown. In the case  $m > 1$ , we refer to (Yangari, 2015) in which is studied the existence and uniqueness of mild solutions of a reaction diffusion system with infinitesimal generators.

In order to improve the notation, we consider the system

$$\begin{cases} \partial_t u + Au &= f(t, u), \quad \forall t \in (0, \infty) \\ u(0) &= u_0, \end{cases} \quad (1)$$

where  $u = (u_i)_{i=1}^m$ ,  $f = (f_i)_{i=1}^m$  and  $A = \text{diag}(A_1, \dots, A_m)$ . Moreover, we consider the norm  $\|u\|_\alpha = \sum_{i=1}^m \|u_i\|_{\alpha,i}$  on the space  $X^\alpha = \prod_{i=1}^m X_i^\alpha$ , where the space  $X_i^\alpha$  is defined in the next section.

Throughout this paper, we assume  $\Omega$  some open set in  $\mathbb{R} \times X^\alpha$  and  $f_i : \Omega \mapsto X$  is locally H older continuous in  $t$  and locally Lipschitz continuous in  $u$  on  $\Omega$  for all  $i \in \llbracket 1, m \rrbracket$ . More

miguel.yangari@epn.edu.ec  
Received: 12/06/2015  
Accepted: 23/05/2016  
Published: 30/09/2016

precisely, if  $(t_1, x_1) \in \Omega$ , there exists a neighborhood  $V \subset \Omega$  such that for  $(t, x) \in V, (s, y) \in V$

$$\|f_i(t, x) - f_i(s, y)\| \leq L_i(|t - s|^{\theta_i} + \|x - y\|_\alpha) \quad (2)$$

for some constants  $L_i > 0$  and without loss of generality we assume  $\theta := \theta_i > 0$  for all  $i \in \llbracket 1, m \rrbracket$ . Moreover, a solution of the initial value system (1) on  $(t_0, t_1)$  is a continuous function  $u : [t_0, t_1] \rightarrow X^m$  such that  $u(t_0) = u_0$  and on  $(t_0, t_1)$  we have  $(t, u(t)) \in \Omega, u(t) \in \mathcal{D}(A), \frac{\partial u}{\partial t}(t)$  exists,  $t \mapsto f(t, u(t))$  is locally Hölder continuous and

$$\int_{t_0}^{t_0+\rho} \|f(t, u(t))\|_m dt < \infty$$

for some  $\rho > 0$  and the differential equation (1) is verified.

## 2. SECTORIAL OPERATORS

Taking  $i \in \llbracket 1, m \rrbracket$  fixed throughout this section, we call a linear operator  $A_i$  in a Banach space  $X$ , a sectorial operator if it is a closed densely defined operator such that, for some  $\phi \in (0, \pi/2), M \geq 1$  and a real  $\varepsilon$ , the sector

$$S_{\varepsilon, \phi} = \{\lambda \mid \phi \leq |\arg(\lambda - \varepsilon)| \leq \pi, \lambda \neq \varepsilon\}$$

is in the resolvent set of  $A_i$  and

$$\|(\lambda I - A_i)^{-1}\| \leq M/|\lambda - \varepsilon| \quad \text{for all } \lambda \in S_{\varepsilon, \phi}.$$

Let us note that every bounded linear operator on a Banach space is sectorial. Also, if we define

$$e^{-A_i t} = \frac{1}{2i\pi} \int_{\Gamma} (\lambda I + A_i)^{-1} e^{\lambda t} d\lambda$$

where  $\Gamma$  is a contour in the resolvent of  $-A_i$  with  $\arg(\lambda) \rightarrow \pm\theta$  as  $|\lambda| \rightarrow \infty$  for some  $\theta \in (\pi/2, \pi)$ , we have that  $-A_i$  is the infinitesimal generator of the analytic semigroup  $(e^{-A_i(t)})_{t \geq 0}$ , moreover, if  $\text{Re}\sigma(A_i) > b_i$ , then for  $t > 0$

$$\|e^{-A_i t}\| \leq ce^{-b_i t}, \quad \|A_i e^{-A_i t}\| \leq \frac{c}{t} e^{-b_i t}. \quad (3)$$

It is important to note that, if  $B$  is a bounded linear operator, then  $e^{-Bt}$  as defined above extends to a group of linear operators and verifies

$$e^{-Bt} e^{-Bs} = e^{-B(t+s)}, \quad \text{for } -\infty < t, s < \infty.$$

In order to define the fractional power of a sectorial operator  $A_i$ , we assume  $\text{Re}\sigma(A_i) > 0$ , so, for any  $\alpha \in (0, 1)$

$$A_i^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-A_i t} dt.$$

Taking  $A_i^{-\alpha}$  defined as above, we have that this operator is a bounded linear operator on  $X$  which is one-one and satisfies  $A_i^{-\alpha} A_i^{-\beta} = A_i^{-(\alpha+\beta)}$ . Furthermore,  $A_i^\alpha$  represents the inverse operator of  $A_i^{-\alpha}$  with  $\mathcal{D}(A_i^\alpha) = R(A_i^{-\alpha})$  and  $A_i^0$  is the identity on  $X$ . An important result concerning positive powers of sectorial operators is

$$\|A_i^\alpha e^{-A_i t}\| \leq c_\alpha t^{-\alpha} e^{-b_i t} \quad \text{for } t > 0 \quad (4)$$

with  $\text{Re}\sigma(A_i) > b_i > 0$  and if  $u \in \mathcal{D}(A_i^\alpha)$

$$\|(e^{-A_i t} - I)u\| \leq \frac{1}{\alpha} c_{1-\alpha} t^\alpha \|A_i^\alpha u\| \quad (5)$$

also,  $A_i^\alpha A_i^\beta = A_i^\beta A_i^\alpha = A_i^{\alpha+\beta}$  on  $\mathcal{D}(A_i^\gamma)$  with  $\gamma = \max(\alpha, \beta, \alpha + \beta)$ .

Now, we consider the fractional powers of  $B_i := A_i + a_i I$  with  $a_i \in \mathbb{R}$  chosen so  $\text{Re}\sigma(B_i) > 0$ , where  $\sigma(B_i)$  is the spectrum of  $B_i$ . We define the Banach space  $X_i^\alpha = \mathcal{D}(B_i^\alpha)$  with the norm  $\|u\|_{\alpha, i} = \|B_i^\alpha u\|$ , where  $\mathcal{D}(B_i^\alpha)$  is the domain of the operator  $B_i^\alpha$ . Finally, taking  $\alpha \geq \beta \geq 0$ , then  $X_i^\alpha$  is a dense subspace of  $X_i^\beta$  with continuous inclusion, also,  $X_i^0 = X$ .

For more information about sectorial operators we refer the reader to (Henry, 1981).

## 3. MAIN RESULTS

In order to state our first result, since  $-A_i$  is the infinitesimal generator of the analytic semigroup  $(e^{-A_i(t)})_{t \geq 0}$  for each  $i \in \llbracket 1, m \rrbracket$ , we define the weak formulation for the system (1) given by

$$u(t) = P(t - t_0)u_0 + \int_{t_0}^t P(t - s)f(s, u(s))ds \quad (6)$$

with

$$P(t) = \text{diag}(e^{-A_1(t)}, \dots, e^{-A_m(t)}).$$

In what follows, the constant  $C > 0$  represents different constants.

**Lemma 3.1** *If  $u$  is the solution of the system (1) on  $(t_0, t_1)$ , then equation (6) is satisfied. Inversely, if  $u$  is a continuous function of  $(t_0, t_1)$  into  $X^\alpha$ ,*

$$\int_{t_0}^{t_0+\rho} \|f(s, u(s))\|_m ds < \infty$$

for some  $\rho > 0$  and equation (6) is satisfied for  $t_0 < t < t_1$ , then  $u$  is a solution of the system (1) on  $(t_0, t_1)$ .

**Proof.** Let assume that  $u$  is the solution of the system (1) on  $(t_0, t_1)$ , taking  $i \in \llbracket 1, m \rrbracket$  fixed, we define the auxiliary function

$$g_i(t, v) = f_i(t, u_1, \dots, u_{i-1}, v, u_{i+1}, \dots, u_n).$$

Let see that  $g_i(t, u_i(t))$  is locally Hölder continuous in  $t$  and  $\int_{t_0}^{t_0+\rho} \|g_i(t, u_i(t))\| dt < +\infty$ . Indeed, since  $g_i : (t_0, t_1) \times X_i^\alpha \rightarrow X$  and

$$\begin{aligned} \|g_i(t, u_i(t)) - g_i(s, u_i(s))\| &= \|f_i(t, u(t)) - f_i(s, u(s))\| \\ &\leq \|f(t, u(t)) - f(s, u(s))\|_m \\ &\leq L|t - s|^\nu \end{aligned}$$

since  $t \rightarrow f(t, u(t))$  is Hölder continuous with exponent  $\nu \in (0, 1)$ . Furthermore,

$$\begin{aligned} \int_{t_0}^{t_0+\rho} \|g_i(t, u_i(t))\|_X dt &= \int_{t_0}^{t_0+\rho} \|f_i(t, u(t))\|_X dt \\ &\leq \int_{t_0}^{t_0+\rho} \|f(t, u(t))\|_m dt \\ &< +\infty \end{aligned}$$

for some  $\rho > 0$ . Now, since  $u$  verifies the system (1), we have that

$$\begin{cases} \partial_t u_i + A_i u_i &= g_i(t, u_i), \\ u_i(0) &= u_{0i}. \end{cases} \quad (7)$$

Therefore by the theorem 3.2.2 in (Henry, 1981), we have that  $u_i$  is the unique solution of the system (7), which can be written as

$$u_i(t) = e^{-A_i(t-t_0)} u_{0i} + \int_{t_0}^t e^{-A_i(t-s)} g_i(s, u_i(s)) ds.$$

Repeating the same procedure for all  $i \in \llbracket 1, m \rrbracket$ , we have

$$u(t) = P(t - t_0) u_0 + \int_{t_0}^t P(t - s) f(s, u(s)) ds$$

namely  $u$  satisfy the equation (6).

Reciprocally, we suppose now that  $u$  satisfy the equation (6) and  $u \in \mathcal{C}((t_0, t_1); X^\alpha)$ . Besides, for each  $i \in \llbracket 1, m \rrbracket$ , we have that  $u_i : (t_0, t_1) \rightarrow X_i^\alpha$  is continuous and verifies

$$u_i(t) = e^{-A_i(t-t_0)} u_{0i} + \int_{t_0}^t e^{-A_i(t-s)} g_i(s, u_i(s)) ds.$$

First, we will prove that  $u_i$  is locally Hölder continuous from  $(t_0, t_1)$  to  $X_i^\alpha$ . Thus, if  $t, t+h \in [t_0^*, t_1^*] \subset (t_0, t_1)$  with  $h > 0$  and  $\delta_i \in (0, 1 - \alpha)$ , we claim that

$$\|u_i(t+h) - u_i(t)\|_{\alpha,i} \leq C_i h^{\delta_i} \quad (8)$$

for some positive constant  $C_i$ . Indeed,

$$\begin{aligned} u_i(t+h) - u_i(t) &= (e^{-A_i h} - I) e^{-A_i(t-t_0)} u_{0i} \\ &\quad + \int_{t_0}^t (e^{-A_i h} - I) e^{-A_i(t-s)} g_i(s, u_i(s)) ds \\ &\quad + \int_t^{t+h} e^{-A_i(t+h-s)} g_i(s, u_i(s)) ds. \end{aligned} \quad (9)$$

Now, for any  $z \in X$ , by Theorem 1.4.3 in (Henry, 1981),

$$\|(e^{-A_i h} - I) e^{-A_i(t-s)} z\|_{\alpha,i} \leq C(t-s)^{-(\alpha+\delta_i)} h^{\delta_i} e^{b_i(t-s)} \|z\|.$$

Moreover, due to each  $f_i$  is locally Hölder in  $t$  and locally Lipschitz in  $u$ , we have that

$$\|f_i(t, u(t)) - f_i(t_0, u(t_0))\| \leq L_i(|t - t_0|^\theta + \|u(t) - u(t_0)\|_\alpha)$$

or equivalently

$$\|g_i(t, u_i(t)) - g_i(t_0, u_i(t_0))\| \leq L_i(|t - t_0|^\theta + \|u(t)\|_\alpha + \|u(t_0)\|_\alpha). \quad (10)$$

But for hypothesis, we know that  $u : (t_0, t_1) \rightarrow X^\alpha$  is continuous, then, we have

$$C_\alpha = \|u(t_0)\|_\alpha + \max_{t_0^* \leq t \leq t_1^*} \|u\|_\alpha < +\infty.$$

Hence, by the inequality (10)

$$\begin{aligned} \|g_i(t, u_i(t))\| &\leq L(|t - t_0|^\theta + 2C_\alpha) + \|g_i(t_0, u_{0i})\| \\ &\leq L(|t - t_0|^\theta + c). \end{aligned}$$

We begin bounding the first term of the equation (9), thus

$$\begin{aligned} \|(e^{-A_i h} - I) e^{-A_i(t-t_0)} u_{0i}\|_{\alpha,i} &\leq C(t_0^* - t_0)^{-(\alpha+\delta_i)} h^{\delta_i} e^{b_i(t_1-t_0)} \|u_{0i}\| \\ &\leq Ch^{\delta_i}, \end{aligned}$$

since  $t \in [t_0^*, t_1^*] \subset (t_0, t_1)$ . Now, let us bound the second term of the equation (9)

$$\begin{aligned} \left\| \int_{t_0}^t (e^{-A_i h} - I) e^{-A_i(t-s)} g_i(s, u_i(s)) ds \right\|_{\alpha,i} &\leq \int_{t_0}^t C(t-s)^{-(\alpha+\delta_i)} h^{\delta_i} e^{b_i(t-s)} \|g_i(s, u_i(s))\| ds \\ &\leq Ch^{\delta_i} \int_{t_0}^t (t-s)^{-(\alpha+\delta_i)} e^{b_i(t-s)} (L|t - t_0|^\theta + c) ds \\ &\leq Ch^{\delta_i} \int_{t_0}^t (t-s)^{-(\alpha+\delta_i)} ds \end{aligned}$$

$$\leq Ch^{\delta_i}.$$

Bounding now the third term of the equation (9), taking  $Re \sigma(B_i) > \gamma_i > 0$  and  $Re(\sigma(-a_i I)) \geq -(a_i + \gamma_i)$ , using inequalities (3) and (4), we have

$$\begin{aligned} & \left\| \int_t^{t+h} e^{-A_i(t+h-s)} g_i(s, u_i(s)) ds \right\|_{\alpha, i} \\ &= \int_t^{t+h} \left\| B_i^\alpha e^{-B_i(t+h-s)} \right\| \\ & \quad \left\| e^{a_i I(t+h-s)} g_i(s, u_i(s)) \right\| ds \\ &\leq \int_t^{t+h} C_\alpha (t+h-s)^{-\alpha} e^{a_i(t+h-s)} \|g_i(s, u_i(s))\| ds \\ &\leq \int_t^{t+h} C_\alpha (t+h-s)^{-\alpha} e^{a_i(t+h-s)} (L|s-t|^\theta + c) ds \\ &\leq c \int_t^{t+h} (t+h-s)^{-\alpha} e^{a_i(t+h-s)} ds \\ &\leq c \int_t^{t+h} (t+h-s)^{-\alpha} e^{-\gamma t_0} e^{-\gamma t_1} e^{-\gamma h} ds \\ &\leq c \int_t^{t+h} (t+h-s)^{-\alpha} ds \\ &\leq ch^{1-\alpha} \\ &\leq Ch^{\delta_i} \end{aligned}$$

for some large enough positive constant  $C$ . Hence, inequality (8) is satisfied. Moreover,  $t \mapsto g_i(t, u_i(t))$  is locally Hölder continuous on  $(t_0, t_1)$ . Indeed,

$$\begin{aligned} & \|g_i(t, u_i(t)) - g_i(s, u_i(s))\| \\ &= \|f_i(t, u_i(t)) - f_i(s, u_i(s))\| \\ &\leq L_i(|t-s|^\theta + \|u(t) - u(s)\|_\alpha) \\ &\leq L_i(|t-s|^\theta + \sum_{i=1}^m \|u_i(t) - u_i(s)\|_{\alpha, i}) \\ &\leq L_i(|t-s|^\theta + \sum_{i=1}^m C_i |t-s|^{\delta_i}) \\ &\leq C(|t-s|^\theta + \sum_{i=1}^m |t-s|^{\delta_i}). \end{aligned}$$

Also,

$$\int_{t_0}^{t_0+\rho} \|g_i(t, u_i(t))\| \leq \int_{t_0}^{t_0+\rho} \|f(t, u(t))\|_m dt < +\infty.$$

Then, by Theorem 3.2.2 in (Henry, 1981),  $u_i$  solves the equation

$$\begin{cases} \partial_t u_i + A_i u_i &= g_i(t, u_i(t)) = f_i(t, u(t)) \\ u_i(0) &= u_{0i} \end{cases}$$

for all  $i \in \llbracket 1, m \rrbracket$ .

Now, we are in position to state our main result in which we establish the existence and uniqueness of solutions to the system (1).

**Theorem 3.1** *If  $f_i$  verifies the hypothesis (2) for each  $i \in \llbracket 1, m \rrbracket$ , then for any  $(t_0, u_0) \in \Omega$  there exists  $T = T(t_0, u_0) > 0$  such that the system (1) has an unique solution  $u$  on  $(t_0, t_0 + T)$  with initial condition  $u(t_0) = u_0$ .*

**Proof.** By the previous lemma is enough to find a solution  $u$  of the equation (6). We choose  $\delta > 0, \tau > 0$  such that the set

$$V = \{(t, x) | t_0 \leq t \leq t_0 + \tau, \|x - u_0\|_\alpha \leq \delta\}$$

is contained in  $\Omega$  and

$$\|f_i(t, x) - f_i(t, y)\| \leq L_i \|x - y\|_\alpha \quad (11)$$

for any  $(t, x), (t, y) \in V$ . Moreover, we claim that for all  $i \in \llbracket 1, m \rrbracket$ , there exists a constant  $M_i > 0$  such that

$$\|B_i^\alpha e^{-A_i t}\| \leq M_i t^{-\alpha} e^{a_i t} \quad (12)$$

for all  $t > 0$ . Indeed, taking  $Re \sigma(B_i) > \gamma_i > 0$  and  $Re(\sigma(-a_i I)) \geq -(a_i + \gamma_i)$ , using inequalities (3) and (4)

$$\begin{aligned} \|B_i^\alpha e^{-A_i t}\| &= \|B_i^\alpha e^{-A_i t} e^{-a_i I t} e^{a_i I t}\| \\ &= \|B_i^\alpha e^{-B_i t} e^{a_i I t}\| \\ &\leq \|B_i^\alpha e^{-B_i t}\| \|e^{a_i I t}\| \\ &\leq C_\alpha t^{-\alpha} e^{-\gamma_i t} \|e^{-(a_i I) t}\| \\ &\leq C_\alpha t^{-\alpha} e^{-\gamma_i t} C e^{(a_i + \gamma_i) t} \\ &\leq M_i t^{-\alpha} e^{a_i t}. \end{aligned}$$

Furthermore, we set

$$\kappa = \max_{[t_0, t_0 + \tau]} \|f_i(t, u_0)\|, \quad L = \max_{i \in \llbracket 1, m \rrbracket} L_i$$

and

$$a = \max_{i \in \llbracket 1, m \rrbracket} a_i, \quad M = \max_{i \in \llbracket 1, m \rrbracket} M_i.$$

Hence, we can choose  $T \in (0, \tau)$  such that

$$\|(e^{-A_i h} - I)u_{0i}\|_{\alpha, i} \leq \frac{\delta}{2m}, \quad \text{for } 0 \leq h \leq T$$

for all  $i \in \llbracket 1, m \rrbracket$  and

$$mM(\kappa + L\delta) \int_0^T u^{-\alpha} e^{au} du \leq \frac{\delta}{2}.$$

If  $S$  denote the set of continuous functions  $y : [t_0, t_0 + T] \rightarrow X^\alpha$  such that  $\|y(t) - u_0\|_\alpha \leq \delta$  on  $t_0 \leq t \leq t_0 + T$ , provided by the norm

$$\|y\|^T = \sum_{i=1}^m \sup\{\|y_i(t)\|_{\alpha,i}, \quad t_0 \leq t \leq t_0 + T\}$$

then  $S$  is a complete metric space since it is the product of complete metric spaces with the product norm. So, for  $y \in S$ , we define  $H(y) : [t_0, t_0 + T] \rightarrow X^m$  given by

$$H(y)(t) = P(t - t_0)u_0 + \int_{t_0}^t P(t - s)f(s, y(s))ds.$$

We claim that  $H : S \rightarrow S$  is a contraction. Indeed, if  $Y \in S$  and  $t_0 \leq t \leq t_0 + T$ , we have that

$$\begin{aligned} \|H(y)(t) - u_0\|_\alpha &\leq \|(P(t - t_0) - I)u_0\|_\alpha \\ &+ \int_{t_0}^t \|P(t - s)f(s, y(s))\|_\alpha ds \\ &= \sum_{i=1}^m \|(e^{-A_i(t-t_0)} - I)u_{0i}\|_{\alpha,i} \\ &+ \sum_{i=1}^m \int_{t_0}^t \|e^{-A_i(t-s)} f_i(s, y(s))\|_{\alpha,i} ds \\ &\leq \frac{\alpha}{2} + \sum_{i=1}^m \int_{t_0}^t \|(A_i)_1^\alpha e^{-A_i(t-s)}\| \|f_i(s, y(s))\| ds \\ &\leq \frac{\delta}{2} + mM \int_{t_0}^t (t - s)^{-\alpha} e^{a(t-s)} (L \|y(s) - u_0\|_\alpha + \kappa) ds \\ &\leq \frac{\delta}{2} + mM(\kappa + L\delta) \int_0^\tau u^{-\alpha} e^{au} du \leq \delta. \end{aligned}$$

We prove now that  $H(y) : [t_0, t_0 + T] \rightarrow X^\alpha$  is continuous. Indeed, without loss of generality, we suppose that  $z < t$ , thus

$$\begin{aligned} \|H(y)(t) - H(y)(z)\|_\alpha &= \sum_{i=1}^m \|H_i(y)(t) - H_i(y)(z)\|_{\alpha,i} \\ &\leq \sum_{i=1}^m \left[ \|(e^{-A_i(t-t_0)} - e^{-A_i(z-t_0)})u_{0i}\|_{\alpha,i} \right. \\ &\quad + \left\| \int_{t_0}^z (e^{-A_i(t-s)} - e^{-A_i(z-s)}) f_i(s, y(s)) ds \right\|_{\alpha,i} \\ &\quad + \left\| \int_z^t e^{-A_i(t-s)} f_i(s, y(s)) ds \right\|_{\alpha,i} \Big] \\ &= \sum_{i=1}^m (I_1 + I_2 + I_3). \end{aligned}$$

Let  $\epsilon > 0$ , we need to find  $\delta > 0$  such that  $|t - z| < \delta$  implies  $\|H(y)(t) - H(y)(z)\|_\alpha < \epsilon$ . Thus, we bound each term of the last inequality

$$\begin{aligned} I_1 &= \|(e^{-A_i(t-t_0)} - e^{-A_i(z-t_0)})u_{0i}\|_{\alpha,i} \\ &= \|B_i^\alpha e^{-A_i(z-t_0)} (e^{-A_i(t-z)} u_{0i} - u_{0i})\| \\ &\leq \|B_i^\alpha\| \|e^{-A_i(z-t_0)}\| \|e^{-A_i(t-z)} u_{0i} - u_{0i}\| \\ &\leq \|B_i^\alpha\| c_i e^{-b_i(z-t_0)} \|e^{-A_i(t-z)} u_{0i} - u_{0i}\| \\ &\leq C_i \|e^{-A_i(t-z)} u_{0i} - u_{0i}\| < \frac{\epsilon}{3m} \end{aligned}$$

the last inequality is satisfied if  $|t - z| < \delta_1$  for some  $\delta_1 > 0$ . Now, bounding  $I_2$

$$\begin{aligned} I_2 &= \left\| \int_{t_0}^z (e^{-A_i(t-s)} - e^{-A_i(z-s)}) f_i(s, y(s)) ds \right\|_{\alpha,i} \\ &\leq \int_{t_0}^z \|B_i^\alpha (e^{-A_i(t-s)} - e^{-A_i(z-s)}) f_i(s, y(s))\| ds \\ &\leq \|B_i^\alpha\| \int_{t_0}^z \|(e^{-A_i(t-z)} - I) e^{-A_i(z-s)} f_i(s, y(s))\| ds \\ &\leq \|B_i^\alpha\| C \int_{t_0}^z \|A_i^\alpha e^{-A_i(z-s)} f_i(s, y(s))\| ds (t - z) \\ &\leq C(t - z) \int_{t_0}^z C_\alpha (z - s)^{-\alpha} e^{-b_i(z-s)} ds \\ &\leq \frac{\epsilon}{3m} \end{aligned}$$

the last inequality is satisfied if  $|t - z| < \delta_2$  for some  $\delta_2 > 0$ . Now, proceeding with  $I_3$ ,

$$\begin{aligned} I_3 &= \left\| \int_z^t e^{-A_i(t-s)} f_i(s, y(s)) ds \right\|_{\alpha,i} \\ &\leq \|B_i^\alpha\| \int_z^t \|e^{-A_i(t-s)}\| \|f_i(s, y(s))\| ds \\ &\leq C \int_z^t e^{-b_i(t-s)} ds \\ &\leq \frac{\epsilon}{3m} \end{aligned}$$

with  $|t - z| < \delta_3$  for some  $\delta_3 > 0$ . Therefore, taking  $\delta = \min\{\delta_1, \delta_2, \delta_3\}$  we conclude that  $H(y) \in \mathcal{C}([t_0, t_0 + T]; X^\alpha)$  and then  $H(y) \in S$ .

We now prove that  $H$  is a contraction. Let  $y, z \in S$  and  $t_0 \leq t \leq t_0 + \tau$

$$\begin{aligned} \|H_i(y)(t) - H_i(z)(t)\|_{\alpha,i} &\leq \int_{t_0}^t \|B_i^\alpha e^{-A_i(t-s)}\| \|f_i(s, y(s)) - f_i(s, z(s))\| ds \\ &\leq \int_{t_0}^t M_i (t - s)^{-\alpha} e^{a_i(t-s)} L_i \|y(s) - z(s)\|_\alpha ds \\ &= ML \int_{t_0}^t (t - s)^{-\alpha} e^{a_i(t-s)} ds \|y - z\|^T \end{aligned}$$

$$\leq ML \int_0^T u^{-\alpha} e^{au} ds \|y - z\|^T$$

therefore, with  $I = [t_0, t_0 + \tau]$

$$\sup_{t \in I} \|H_i(y)(t) - H_i(z)(t)\|_{\alpha,i} \leq ML \int_0^T u^{-\alpha} e^{au} ds \|y - z\|^T$$

for all  $i \in \llbracket 1, m \rrbracket$ . Hence

$$\begin{aligned} \|H(y)(t) - H(z)(t)\|^T &\leq mML \int_0^T u^{-\alpha} e^{au} ds \|y - z\|^T \\ &\leq \frac{1}{2} \|y - z\|^T. \end{aligned}$$

For the fixed point theorem,  $H$  has a unique solution  $u \in S$ , where  $u$  is a continuous function  $u : [t_0, t_0 + T] \rightarrow (x_i^\alpha)^m$  which verifies the equation (6). Also,  $f_i$  satisfies

$$\|f_i(t, u) - f_i(t_0, u_0)\| \leq L_i(|t - t_0|^\theta + \|u - u_0\|_\alpha)$$

thus,

$$\begin{aligned} \|f_i(t, u)\| &\leq L_i(|t - t_0|^\theta + \delta) + \|f_i(t_0, u_0)\| \\ &\leq L(T^\theta + \delta) + B \leq C \end{aligned}$$

and then, for any fixed  $\rho > 0$

$$\begin{aligned} \int_{t_0}^{t_0+\rho} \|f(s, u(s))\|_m ds &= \sum_{i=1}^m \int_{t_0}^{t_0+\rho} \|f_i(s, u(s))\| ds \\ &\leq mC_2\rho < +\infty. \end{aligned}$$

Therefore, for the Lemma 3.1,  $u$  is solution of the system (1) on  $(t_0, t_0 + T)$ .

As a consequence of the previous theorem, we present a result about the behavior of the solution.

**Theorem 3.2** *If  $f_i$  verifies the hypothesis (2) for each  $i \in \llbracket 1, m \rrbracket$  and for all closed and bounded subset  $V \subset \Omega$  the image of  $f(V)$  is bounded in  $X^m$ . Then  $u$  is a solution of the system (1) on  $(t_0, t_1)$  and  $t_1$  is maximal, i.e., there is no solution of the system (1) on  $(t_0, t_2)$  if  $t_2 > t_1$ , then either  $t_1 = +\infty$  or else there exists a sequence  $t_n \rightarrow t_1^-$  as  $n \rightarrow +\infty$  such that  $(t_n, u(t_n)) \rightarrow \partial\Omega$ . If  $\Omega$  is unbounded, the point at infinity is included in  $\partial\Omega$ .*

**Proof.** Proceeding by contradiction, we suppose that  $t_1 < +\infty$ , but  $(t, u(t))$  is not in a neighborhood  $N$  of  $\partial\Omega$  for  $\underline{t} \leq t < t_1$ , we can take  $N$  of the form  $N = \Omega \setminus B$ , where  $B$  is a closed and bounded set of  $\Omega$  and  $(t, u(t)) \in B$  for all  $\underline{t} \leq t < t_1$ . We will prove that there exists  $x_1 \in X^\alpha$  such that  $(t_1, x_1) \in B$  with  $u(t) \rightarrow x_1$  in  $X^\alpha$  when  $t \rightarrow t_1^-$ . Even more  $u(t_1) = x_1$ , which means that the solution could be extended until  $t_1$ . Indeed, let  $C = \sup\{\|f(t, u)\|_m, (t, u) \in B\}$ , so  $C < +\infty$

because  $B$  is bounded and closed and by hypothesis  $f(B)$  is bounded in  $X^m$ .

Firstly, we can see that if  $\alpha \leq \beta < 1$  and  $\underline{t} \leq t < t_1$ , then

$$\begin{aligned} \|u(t)\|_\beta &= \sum_{i=1}^m \|u_i(t)\|_{x_i^\beta} \\ &\leq \sum_{i=1}^m \left[ \left\| e^{-A_i(t-t_0)} u_{0i} \right\|_{\beta,i} \right. \\ &\quad \left. + \int_{t_0}^t \left\| e^{-A_i(t-s)} f_i(s, u(s)) \right\|_{\beta,i} ds \right] \\ &= \sum_{i=1}^m \left[ \left\| B_i^\alpha B_i^{\beta-\alpha} e^{-A_i(t-t_0)} u_{0i} \right\| \right. \\ &\quad \left. + \int_{t_0}^t \left\| B_i^\beta e^{-A_i(t-s)} \right\| \|f_i(s, u(s))\| ds \right] \\ &\leq \sum_{i=1}^m c \|B_i^\alpha\| \left\| B_i^{\beta-\alpha} e^{-A_i(t-t_0)} \right\| \|u_{0i}\|_{\alpha,i} \\ &\quad + \int_{t_0}^t \left\| B_i^\beta e^{-A_i(t-s)} \right\| \|f_i(s, u(s))\| ds \\ &\leq c \sum_{i=1}^m \left[ M_i(t-t_0)^{-(\beta-\alpha)} e^{a_i(t-t_0)} \|u_{0i}\|_{\alpha,i} \right. \\ &\quad \left. + \int_{t_0}^t M_i(t-s)^{-\beta} e^{a_i(t-s)} \|f_i(s, u(s))\| ds \right] \\ &\leq c \left[ (t-t_0)^{-(\beta-\alpha)} + \int_{t_0}^t (t-s)^{-\beta} ds \right] \\ &\leq C. \end{aligned}$$

Thus,  $\|u(t)\|_\beta$  remains bounded when  $t \rightarrow t_1^-$ . Now, we suppose that  $\underline{t} \leq \tau < t < t_1$ , so

$$u(t) - u(\tau) = (P(t-\tau) - I)u(\tau) + \int_\tau^t P(t-s)f(s, u(s))ds,$$

then

$$\begin{aligned} \|u(t) - u(\tau)\|_{\alpha,i} &\leq \sum_{i=1}^m \left[ c \left\| (e^{-A_i(t-\tau)} - I)u_i(\tau) \right\|_{\beta,i} \right. \\ &\quad \left. + \int_\tau^t \left\| e^{-A_i(t-s)} f_i(s, u(s)) \right\|_{\alpha,i} ds \right] \\ &\leq \sum_{i=1}^m \left[ c \|B_i^\alpha\| \left\| B_i^{\beta-\alpha} (e^{-A_i(t-\tau)} - I)u_i(\tau) \right\| \right. \\ &\quad \left. + \int_\tau^t \left\| B_i^\alpha e^{-A_i(t-s)} f_i(s, u(s)) \right\| ds \right]. \end{aligned}$$

Bounding the first term, let  $\epsilon_1 > 0$  an arbitrary number. Since  $\mathcal{D}(A_i^{\beta-\alpha})$  is dense in  $X$ , we take  $v \in \mathcal{D}(A_i^{\beta-\alpha})$  such that  $\|u_i(\tau) - v\| < \eta$ , thus

$$\begin{aligned}
 & \left\| B_i^{\beta-\alpha} (e^{-A_i(t-\tau)} - I) u_i(\tau) \right\| \\
 & \leq \left\| B_i^{\beta-\alpha} \right\| \left\| (e^{-A_i(t-\tau)} - I) u_i(\tau) \right\| \\
 & \leq c \left\| (e^{-A_i(t-\tau)} - I) (u_i(\tau) - v) + (e^{-A_i(t-\tau)} - I) v \right\| \\
 & \leq c \left[ \left\| e^{-A_i(t-\tau)} (u_i(\tau) - v) \right\| + \|u_i(\tau) - v\| \right. \\
 & \quad \left. + \left\| (e^{-A_i(t-\tau)} - I) v \right\| \right] \\
 & \leq c \left[ e^{-b_i(t-\tau)} \|u_i(\tau) - v\| + \|u_i(\tau) - v\| \right. \\
 & \quad \left. + \left\| (e^{-A_i(t-\tau)} - I) v \right\| \right] \\
 & \leq c \left[ e^{-b_i t_1} \eta + \eta + (t - \tau)^{\beta-\alpha} \left\| A_i^{\beta-\alpha} v \right\| \right] \\
 & \leq \epsilon_1 + C(t - \tau)^{\beta-\alpha}.
 \end{aligned}$$

When the last bound is given when  $\eta$  is small enough such that  $c(e^{-a_i(t-\tau)}\eta + \eta) < \epsilon_1$ . Now, bounding the second term

$$\begin{aligned}
 & \int_{\tau}^t \left\| B_i^{\alpha} e^{-A_i(t-s)} f_i(s, u(s)) \right\| ds \\
 & \leq c \int_{\tau}^t \left\| B_i^{\alpha} e^{-A_i(t-s)} \right\| \|f_i(s, u(s))\| ds \\
 & \leq c M_i \int_{\tau}^t (t-s)^{-\alpha} e^{a_i(t-s)} ds \\
 & \leq C(t-s)^{1-\alpha}.
 \end{aligned}$$

Thus, we have

$$\|u(t) - u(\tau)\|_{\alpha} \leq C \left[ \epsilon_1 + (t - \tau)^{\beta-\alpha} + (t - \tau)^{1-\alpha} \right].$$

Now, we consider  $t_n \rightarrow t_1^-$  and let define  $u_n = u(t_n)$ , thus, taking  $\epsilon > 0$ , we have

$$\begin{aligned}
 \|u_n - u_m\|_{\alpha} & \leq C \left[ \epsilon_1 + (t_n - t_m)^{\beta-\alpha} + (t_n - t_m)^{1-\alpha} \right] \\
 & < \epsilon.
 \end{aligned}$$

if  $n, m$  are large enough. Therefore,  $(u_n)$  is a Cauchy sequence in the complete space  $X^{\alpha}$ , thus, there exists  $x_1 \in X^{\alpha}$  such that  $u_n \rightarrow x_1$ , it means  $(t_n, u(t_n)) \rightarrow (t_1, x_1)$  and since  $(t_n, u(t_n)) \in B$  with  $B$  closed, we can conclude that  $(t_1, x_1) \in B$ .

Also, since  $u : (t_0, t_1) \rightarrow X^m$  is continuous and  $t_n \rightarrow t_1^-$ , we have that  $(t_n, u(t_n)) \rightarrow (t_1, u(t_1)) \in B$ , thus, we can conclude that  $u(t_1) = x_1$ .

To finish the proof, using the Theorem 3.1 and considering  $(t_1, x_1) \in B \subset \Omega$ , we can find an unique solution  $v$  on

$(t_1, t_1 + T(t_1))$  for some  $T(t_1) > 0$  of the system (1) with initial condition  $v(t_1) = x_1$ . Hence, taking

$$z(t) = \begin{cases} u(t) & \text{if } t \in [t_0, t_1] \\ v(t) & \text{if } t \in [t_1, t_1 + T(t_1)] \end{cases}$$

we note that  $z$  is continuous in  $[t_0, t_1 + T(t_1)]$ . So, we conclude that  $z$  is a solution of the system (1) with  $z(t_0) = x_0$  on  $(t_0, t_1 + T(t_1))$  which contradict the maximality of  $t_1$ .

Finally, we state that under some extra conditions the unique solution is global in time.

**Theorem 3.3** *Let us suppose that  $\Omega = (\tau, +\infty) \times X^{\alpha}$  and  $f_i(t, x)$  satisfies hypothesis (2) for each  $i \in \llbracket 1, m \rrbracket$ . Furthermore, there exists  $k(\cdot)$  a continuous function on  $(\tau, +\infty)$  that verifies*

$$\|f(t, u)\|_m \leq k(t)(1 + \|u\|_{\alpha})$$

for all  $(t, u) \in \Omega$ . If  $t_0 > \tau$ ,  $u_0 \in X^{\alpha}$ , the unique solution of the system (1) with  $u(t_0) = u_0$  exists for all  $t > t_0$ .

**Proof.** Firstly, we can note that hypothesis of the Theorem 3.2 are satisfied. Proceeding by contradiction, we take  $t_0 > \tau$  and assume that there exists a unique solution of the system (1) defined in  $(t_0, t_1)$  where  $t_1$  is maximal, so, for the last result exists a sequence  $t_n \rightarrow t_1^-$  such that  $\|u(t_n)\|_{\alpha} \rightarrow +\infty$ . However, since  $\beta < \alpha$  implies  $X_i^{\alpha} < X_i^{\beta}$  for all  $i \in \llbracket 1, m \rrbracket$ , taking  $t \in (t_1, t_1)$ , by a similar procedure to the previous theorem and since  $K(\cdot)$  is continuous on  $(\tau, \infty)$ , i.e., bounded on  $[t_0, t_1]$ , we have

$$\begin{aligned}
 \|u(t)\|_{\alpha} & \leq C \left[ (t - t_0)^{-(\alpha-\beta)} \right. \\
 & \quad \left. + \int_{t_0}^t (t-s)^{-\alpha} \|f(s, u(s))\|_m ds \right] \\
 & \leq c \left[ (t - t_0)^{-(\alpha-\beta)} \right. \\
 & \quad \left. + \int_{t_0}^t (t-s)^{-\alpha} k(s)(1 + \|u(s)\|_{\alpha} ds \right] \\
 & \leq c \left[ (t - t_0)^{-(\alpha-\beta)} \right. \\
 & \quad \left. + \int_{t_0}^t (t-s)^{-\alpha} (1 + \|u(s)\|_{\alpha} ds \right] \\
 & \leq c + \int_{t_0}^t c(t-s)^{-\alpha} \|u(s)\|_{\alpha} ds
 \end{aligned}$$

for the Bellman-Gronwall theorem, we can conclude that

$$\|u(t)\|_{\alpha} \leq C e^{\int_{t_0}^t c(t-s)^{-\alpha} ds}$$

$$\leq C \quad \forall t \in (t_1, t_1).$$

Which is a contradiction with the fact that  $\|u(t_n)\|_\alpha \rightarrow +\infty$  when  $t_n \rightarrow t_1^-$ .

#### 4. CONCLUSIONS

Similarly to the problem with a single equation, using the properties and estimations of sectorial operators, we state a general result concerning the existence and uniqueness of solutions to systems of equations, when the diffusion terms are given by sectorial generators, also, assuming additional hypothesis on the forcing term, a result of global existence in time is presented. The computations stated in the paper are based in the application of the Banach Fixed Point Theorem.

#### REFERENCES

Aizicovici S., and McKibben M. (2000). Existence results for a class of abstract nonlocal Cauchy problems. *Nonlinear Analysis*. 39, 649-668.

Benchohra, M and Ntouyas, S. (2001). Nonlocal Cauchy problems for neutral functional differential and integrodifferential inclusions in Banach spaces. *J. Math. Anal. Appl.* 258, 573-590.

Byszewski L. and Lakshmikantham V. (1990). Theorem about the existence and uniqueness of a solutions of a non- local Cauchy problem in a Banach space. *Appl. Anal.* 40, 11-19.

Byszewski, L. (1991). Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. *J. Math. Anal. Appl.* 162, 494-505.

Byszewski, L. (1993). Existence and uniqueness of solutions of semilinear evolution nonlocal Cauchy problem. *Zesz. Nauk. Pol. Rzes. Mat. Fiz.* 18, 109-112.

Byszewski, L. and Akca, H. (1998). Existence of solutions of a semilinear functional-differential evolution non-local problem. *Nonlinear Analysis*. 34, 65-72.

Cabré, X. and Roquejoffre, J. (2013). The influence of fractional diffusion in Fisher-KPP equation. *Commun. Math. Phys.* 320, 679-722.

Fu, X. and Ezzinbi, K. (2003). Existence of solutions for neutral functional differential evolution equations with non- local conditions. *Nonlinear Analysis*. 54, 215-227.

Henry, D. (1981). *Geometric theory of semilinear parabolic equations*. Berlin, Germany: Springer-Verlag.

Jackson, D. (1993). Existence and uniqueness of solutions to semilinear nonlocal parabolic equations. *J. Math. Anal. Appl.* 172, 256-265.

Liang, J., Van Casteren, J., and Xiao, T. (2002). Nonlocal Cauchy problems for semilinear evolution equations. *Nonlinear Anal. Ser. A: Theory Methods*. 50, 173-189.

Lin, Y. and Liu, J. (1996). Semilinear integrodifferential equations with nonlocal Cauchy Problems. *Nonlinear Analysis*. 26, 1023-1033.

Ntouyas, S. and Tsamotas, P. (1997). Global existence for semilinear evolution equations with nonlocal conditions. *J. Math. Anal. Appl.* 210, 679-687.

Ntouyas, S. and Tsamotas, P. (1997). Global existence for semilinear integrodifferential equations with delay and non- local conditions. *J. Math. Anal. Appl.* 64, 99-105.

Yangari, M. (2015). Existence and Uniqueness of Global Mild Solutions for Nonlocal Cauchy Systems in Banach Spaces. *Revista Politécnica*. 35(2), 149-152.



Miguel Angel Sosa Yangari. Mathematician graduated from Escuela Politécnica Nacional, Ecuador. Doctor of Engineering Sciences, mention Mathematical Modeling at the Universidad de Chile, Chile. Doctor in Applied Mathematics at the University of Toulouse, France. Currently is Associate Professor, Level 3, Grade 5 at Escuela Politécnica Nacional.



Diego Salazar Israel Orellana. Mathematician graduated from Escuela Politécnica Nacional, Ecuador and graduate M.Sc. Pure Mathematics, UCE-EPN-USFQ. He is currently Ocasional Professor II at Escuela Politécnica Nacional.